Emergence of geometry in quantum mechanics based on finite groups

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Abstract. In the framework of constructive quantum mechanics, we consider the emergence of geometry from entanglement in composite quantum systems. We specify the most general structure of the symmetry group of a quantum system with geometry. We show that the 2nd Rényi entanglement entropy may be useful in applying polynomial computer algebra to model metric structures in quantum systems with geometry.

1. Introduction

In [1, 2, 3] we proposed a constructive modification of quantum mechanics that replaces the unitary group in a Hilbert space over the field $\mathbb{C}$ with the unitary representation of a finite group in a Hilbert space over an abelian extension of $\mathbb{Q}$ which is a dense subfield of $\mathbb{R}$ or $\mathbb{C}$ depending on the structure of the group. T. Banks recently [4] analyzed this modification from the point of view of real physics and cosmology and came to the conclusion that it "can probably be a model of the world we observe."

In short, constructive quantum mechanics boils down to the following. We start with the set $\Omega = \{e_1, \ldots, e_{\mathcal{N}}\} \cong \{1, \ldots, \mathcal{N}\}$ of "types" of primary ("ontic") objects on which a permutation group G acts (T. Banks showed that it suffices to assume that $G = \mathsf{S}_{\mathcal{N}}$ in order to "encompass finite dimensional approximations to all known models of theoretical physics"). Let n_i be the number of instances of ontic objects of the ith type. Then the set of all objects can be described by the vector

$$\left|n\right\rangle = \left(n_1, \ldots, n_{\mathcal{N}}\right)^{\mathsf{T}}. \tag{1}$$

These "ontic" vectors form the semimodule H_Ω over the semiring of natural numbers $\mathbb{N} = \{0, 1, 2, \ldots\}$.

The action of G on Ω determines the *permutation representation* $\mathcal{P}(G)$ in the semimodule H_Ω. For $g \in G$, the matrix of the permutation representation has the form $\mathcal{P}(g)_{i,j} = \delta_{ig,j}$. Using standard mathematical procedures, the semiring $\mathbb{N}$ can be extended to a field $\mathcal{F}$ which is a *splitting field* for the group G. The field $\mathcal{F}$ is a subfield of ℓth *cyclotomic field*, where ℓ is the *exponent* of the group G. Depending on the structure of G, the field $\mathcal{F}$ is a dense subfield of either $\mathbb{R}$ or $\mathbb{C}$, i.e., $\mathcal{F}$ is physically indistinguishable from these continuous fields. The extension of $\mathbb{N}$ to $\mathcal{F}$ induces the extension of the ontic semimodule H_Ω to the

Hilbert space $\mathcal{H}_\Omega$. The inner product in this Hilbert space is a natural extension of the *standard inner product* in the ontic semimodule: $\langle m \mid n\rangle = \sum_{i=1}^{\mathcal{N}} m_i n_i$, where $|m\rangle = (m_1,\ldots,m_{\mathcal{N}})^{\mathsf{T}}$ and $|n\rangle = (n_1,\ldots,n_{\mathcal{N}})^{\mathsf{T}}$ are ontic vectors. The standard inner product is invariant under the representation $\mathcal{P}(G)$)

Since $\mathcal{F}$ is a splitting field, we can decompose the Hilbert space $\mathcal{H}_\Omega$ into irreducible subspaces that are invariant with respect to the representation $\mathcal{P}(G)$:

$$\mathcal{H}_\Omega = \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \cdots \oplus \mathcal{H}_K\,.$$

This decomposition can be constructed algorithmically by calculating the complete set of mutually orthogonal invariant projectors: $B_1, B_2, \ldots, B_K$.[1] An arbitrary invariant subspace $\mathcal{H}_\alpha \leq \mathcal{H}_\Omega$ is a direct sum of irreducible ones:

$$\mathcal{H}_\alpha = \bigoplus_{k'\in\alpha} \mathcal{H}_{k'}, \quad \alpha \subseteq \{1,\ldots,K\}\,.$$

Accordingly, the projection operator in $\mathcal{H}_\alpha$ has the form $B_\alpha = \sum_{k'\in\alpha} B_{k'}$.

In any invariant subspace $\mathcal{H}_\alpha$, an independent quantum system can be constructed, since the results of both unitary evolutions and projective measurements applied to any vector belonging to the subspace $\mathcal{H}_\alpha$ will remain in this subspace.

The inner product for the projections $|\varphi\rangle = B_\alpha\,|m\rangle$ and $|\psi\rangle = B_\alpha\,|n\rangle$ of ontic vectors takes the form $\langle\varphi \mid \psi\rangle_\alpha = \langle m\,|B_\alpha|\,n\rangle$. In terms of ontic vectors, a pure state in the subspace $\mathcal{H}_\alpha$ can be represented as the unit vector $|\psi\rangle = \frac{B_\alpha|n\rangle}{\sqrt{\langle n|B_\alpha|n\rangle}}$ or as the density matrix $\rho = \frac{B_\alpha|n\rangle\langle n|B_\alpha}{\langle n|B_\alpha|n\rangle}$. Operators of unitary evolution in the subspace $\mathcal{H}_\alpha$ have the form $U_{\alpha,g} = B_\alpha\mathcal{P}(g)$.

2. Symmetry Group of Composite Quantum System

The Hilbert space of an N-component quantum system has the form

$$\widetilde{\mathcal{H}} = \bigotimes_{x\in X} \mathcal{H}_x\,. \tag{2}$$

where $X \cong \overline{N} = \{1,\ldots,N\}$. A Hilbert space that can be decomposed into a tensor product of spaces of smaller dimensions is a special case of a general Hilbert space, so it is natural to assume that structures like (2) arise as approximations. This is consistent with the general "holistic" view that the partition of the system as a whole into subsystems is always conditional and approximate.

We make the following assumptions:

- The set X of indices of "local" Hilbert spaces $\mathcal{H}_x$ has symmetries that form the group G.
- The local Hilbert spaces are isomorphic, i.e., $\mathcal{H}_x \cong \mathcal{H}$ for any $x \in X$, where $\mathcal{H}$ is a representative of the equivalence class of spaces $\mathcal{H}_x$.

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[1] We have developed and implemented an efficient algorithm for such calculations [5].

- In the local space $\mathcal{H}$, the unitary representation acts, which is a subrepresentation of the permutation representation of the group F acting on the set $V \cong \overline{M} = \{1, \ldots, M\}$, that is, the set V is the basis of the permutation representation.

The set X can be interpreted as a "geometric space", and the group G as a group of "spatial" symmetries. The group F is interpreted as a group of "local" symmetries.

Based on the natural properties that a geometric space must have, we can show that the group $\widetilde{W}$, which combines spatial and local symmetries, belongs to an equivalence class of group extensions of the form

$$\begin{array}{ccccccc} & & & \widetilde{W} & & & \\ & & \nearrow & & \searrow & & \\ \mathbf{1} \longrightarrow & F^X & & \Phi \big\downarrow & & G & \longrightarrow \mathbf{1} \ , \\ & & \searrow & & \nearrow & & \\ & & & \widetilde{W}' & & & \end{array} \tag{3}$$

where F^X is a group of F-valued functions on the space X, and $\Phi : \widetilde{W} \to \widetilde{W}'$ is a group isomorphism that provides the commutativity of the diagram.

The set of elements of $\widetilde{W}$ can be identified with the Cartesian product of the sets F^X and G, i.e., the elements of $\widetilde{W}$ can be represented as pairs $(f(x)\,, g)$, where $f(x) \in F^X$, $g \in G$. Explicit calculations lead to the following:

- The equivalence classes of extensions (3) are parameterized by *antihomomorphisms* of the space group, that is, by functions $\mu : G \to G$ such that $\mu(ab) = \mu(b)\,\mu(a)$ for any $a, b \in G$.
- An isomorphism of equivalent extensions has the form
$$\Phi : (f(x)\,, g) \mapsto (f(x\varphi(g))\,, g)\,,$$
where $\varphi : G \to G$ is an *arbitrary* function.
- The main group operations have the following explicit form:
$$v(x)\,(f(x)\,, g) = v(x\mu(g))\,f(x\varphi(g))\,, \tag{4}$$
$$(f(x)\,, g)\,(f'(x)\,, g') = \Big(f\Big(x\varphi(gg')^{-1}\mu(g')\,\varphi(g)\Big)\,f'\Big(x\varphi(gg')^{-1}\varphi(g')\Big)\,, gg'\Big)\,, \tag{5}$$
$$(f(x)\,, g)^{-1} = \left(f\Big(x\varphi\big(g^{-1}\big)^{-1}\mu(g)^{-1}\,\varphi(g)\Big)^{-1}, g^{-1}\right), \tag{6}$$
where (4) is the action of $(f(x)\,, g) \in \widetilde{W}$ on the function $v(x) \in V^X$,
(5) is the group multiplication in $\widetilde{W}$, and (6) is the group inversion.

There are two universal (i.e., existing for any group, regardless of its specific properties) antihomomorphisms: $\mu(g) = \mathbf{1}$ and $\mu(g) = g^{-1}$. The choice of $\mu(g) = \mathbf{1}$ leads to the trivial extension, i.e., to the direct product $\widetilde{W} \cong F^X \times G$. The antihomomorphism (in fact, antiisomorphism) $\mu(g) = g^{-1}$ leads to a semidirect product of the groups F^X and G, which is called the *wreath product* of the groups F and G:

$$\widetilde{W} = F \wr G \cong F^X \rtimes G. \tag{7}$$

As for the arbitrary function φ, we use two options in the implementation of our algorithms : $\varphi(g) = g^{-1}$ and $\varphi(g) = \mathbf{1}$. In these cases, expressions (4) – (6) for group operations are more or less compact:

	$\varphi(g) = g^{-1}$	$\varphi(g) = \mathbf{1}$
$v(x)\,(f(x)\,,g) =$	$v\left(xg^{-1}\right) f\left(xg^{-1}\right)$	$v\left(xg^{-1}\right) f(x)$
$(f(x)\,,g)\,(f'(x)\,,g') =$	$(f(x)\,f'(xg)\,,gg')$	$\left(f\left(xg'^{-1}\right) f'(x)\,,gg'\right)$
$(f(x)\,,g)^{-1} =$	$\left(f\left(xg^{-1}\right)^{-1},g^{-1}\right)$	$\left(f(xg)^{-1},g^{-1}\right)$

The unitary representations of the group (7) in the whole Hilbert space (2) describe the quantum properties of the system as a whole. To calculate invariant projectors and decompose permutation representations of wreath products into irreducible components, we developed an algorithm [6], whose C implementation splits representations having dimensions and ranks up to 10^{16} and 10^9, respectively.

3. Emergence of Geometry From Entanglement

The natural idea is to determine the distances between points in the space X in terms of quantum correlations: the greater the correlation, the less the distance. Quantitatively, quantum correlations are described by *measures of entanglement*. The problems of constructing metrics and topology in entangled quantum systems are considered, in particular, in [7, 8, 9].

Denote by $\mathcal{D}\left(\widetilde{\mathcal{H}}\right)$ the set of all states (density matrices) in the Hilbert space (2). The set of *separable states* $\mathcal{D}_{\mathrm{S}}\left(\widetilde{\mathcal{H}}\right)$ consists of states $\rho \in \mathcal{D}\left(\widetilde{\mathcal{H}}\right)$ that can be represented as weighted sums of tensor products of states of components:

$$\rho = \sum_k w_k \bigotimes_{x\in X} \rho_x^k, \quad w_k \geq 0, \quad \sum_k w_k = 1, \quad \rho_x^k \in \mathcal{D}(\mathcal{H}_x)\,.$$

The set of *entangled states* $\mathcal{D}_{\mathrm{E}}\left(\widetilde{\mathcal{H}}\right)$ is defined as the complement of $\mathcal{D}_{\mathrm{S}}\left(\widetilde{\mathcal{H}}\right)$ in the set of all states: $\mathcal{D}_{\mathrm{E}}\left(\widetilde{\mathcal{H}}\right) = \mathcal{D}\left(\widetilde{\mathcal{H}}\right) \setminus \mathcal{D}_{\mathrm{S}}\left(\widetilde{\mathcal{H}}\right)$.

Let ρ_{AB} denote the density matrix for a composite quantum system consisting of components A and B. The statistics of observations of subsystem A are reproduced by the *reduced density matrix* $\rho_A = \mathrm{tr}_B\,(\rho_{AB})$, where the *partial trace* tr_B over subsystem B is defined by the relation

$$\mathrm{tr}_B\left(\left|a_1\right\rangle\left\langle a_2\right| \otimes \left|b_1\right\rangle\left\langle b_2\right|\right) = \left|a_1\right\rangle\left\langle a_2\right| \mathrm{tr}\left(\left|b_1\right\rangle\left\langle b_2\right|\right),$$

which must hold for any vectors $\left|a_1\right\rangle, \left|a_2\right\rangle \in \mathcal{H}_A$ and $\left|b_1\right\rangle, \left|b_2\right\rangle \in \mathcal{H}_B$.

The constructions considered below depend on "the quantum state of the universe"

$$\rho_X \in \widetilde{\mathcal{H}} . \tag{8}$$

There are a variety of entanglement measures [10]. A typical measure of entanglement for the pair of points $\{x, y\} \subseteq X$ is the *mutual information*

$$I(x,y) = S(\rho_x) + S(\rho_y) - S(\rho_{xy}) , \tag{9}$$

where $\rho_x = \operatorname{tr}_{X\setminus\{x\}} \rho_X$, $\rho_y = \operatorname{tr}_{X\setminus\{y\}} \rho_X$, and $\rho_{xy} = \operatorname{tr}_{X\setminus\{x,y\}} \rho_X$. The function $S(\rho)$ is called *entanglement entropy*. The entanglement entropy is usually defined as the *von Neumann entropy* $S(\rho) = -\operatorname{tr}(\rho \log \rho)$, which is the quantum version of the *Shannon entropy*

$$H(p_1, \dots, p_n) = -\sum_{k=1}^{n} p_k \log p_k , \tag{10}$$

where $p_1, \dots, p_n$ is a probability distribution.

From a general point of view, entropy is a function on probability distributions that satisfies some natural postulates. A. Rényi proved [11] that such functions form the following family

$$H_q(p_1, \dots, p_n) = \frac{1}{1-q} \log \sum_{k=1}^{n} p_k^q , \tag{11}$$

where $q \geq 0$ and $q \neq 1$. The function H_q is called the Rényi entropy of order q.

The Shannon entropy (10) is a limiting case of (11): $H \equiv H_1 = \lim_{q\to 1} H_q$. Note that the Shannon entropy has better statistical properties compared to the Renyi entropies with $q \neq 1$, for which, in particular, expression (9) can take negative values. The entropy $H_2(p_1, \dots, p_n) = -\log \sum_{k=1}^{n} p_k^2$ is called the *collision entropy*.

The *quantum Rényi entropy* is the quantum analogue of (11):

$$S_q(\rho) = \frac{1}{1-q} \log \operatorname{tr}(\rho^q) .$$

We will use the 2nd quantum Rényi entropy (quantum collision entropy)

$$S_2(\rho) = -\log \operatorname{tr}\left(\rho^2\right) \tag{12}$$

as the entanglement entropy for the following reasons.

Gleason's theorem provides a one-to-one correspondence between probability measures on subspaces of a Hilbert space and quantum states in this space. More specifically, the most general expression for the Born probability has the form $\mathbb{P} = \operatorname{tr}(\rho_O \rho_S)$, where ρ_O and ρ_S are quantum states of the "observer" and the "observed system", respectively. Since the Born rule is the only fundamental source of probability in quantum theory, it is natural to associate a single state ρ with some Born probability. The probability $\mathbb{P} = \operatorname{tr}\left(\rho^2\right)$ – "the system observes itself" – is such a choice, and its logarithm is precisely the 2nd Rényi entropy (12).

In models of emergent space, the geodesic distance between local quantum subsystems is determined by a certain monotonic function of the entanglement measure [9]. Such a "scaling" function should, at least approximately, tend to zero for maximally entangled pairs of local subsystems, tend to infinity for separable pairs, and satisfy the usual distance properties, such as the triangle inequality, etc. Using the 2nd Rényi entropy as the entanglement entropy, we can get rid of the logarithms in computer algebra calculations by replacing the mutual information (9) with the expression

$$P(x,y)=\exp(-I(x,y))=\frac{\operatorname{tr}\left(\rho_{xy}^2\right)}{\operatorname{tr}\left(\rho_x^2\right)\operatorname{tr}\left(\rho_y^2\right)}\,. \tag{13}$$

For a separable pair $\{x,y\}$, we have $\rho_{xy}=\rho_x\otimes\rho_y$ and, therefore, $P(x,y)=1$. For a maximally entangled pair $P(x,y)=(\dim\mathcal{H})^2$, where $\mathcal{H}$ is the local Hilbert space. $\rho_{xy}\neq\rho_x\otimes\rho_y$ implies $P(x,y)\neq 1$, so expression (13) can quantify the quantum correlation between x and y. For the pure state (8), expression (13) is a combination of polynomials in the coordinates of the ontic vector (1).

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