

Emergence of Geometry in Quantum Mechanics Based on Finite Groups

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Quantum mechanics from permutations of ontic elements

- Ontic¹ elements: $\Omega = \{e_1, \dots, e_N\} \cong \{1, \dots, N\}$
- Natural semimodule $H = \mathbb{N}^N$ consists of natural vectors $|n\rangle = (n_1, \dots, n_N)^T$, $n_i \in \mathbb{N} = \{0, 1, \dots\}$
- Permutation group $G \leq S_N$ that acts transitively on Ω
- Permutation representation of $g \in G$ in H : $\mathcal{P}(g)_{i,j} = \delta_{ig,j}$
- Splitting field for G : $\mathcal{F} \leq \mathbb{Q}\left(e^{i\frac{2\pi}{\ell}}\right)$, ℓ is a divisor of $\text{Exp}(G)$
 \mathcal{F} is a dense subfield of either \mathbb{R} or \mathbb{C}
- $\mathbb{Q}\left(e^{i\frac{2\pi}{\ell}}\right) = \text{Quot}\left(\mathbb{N}\left[e^{i\frac{2\pi}{\ell}}\right]\right) \Rightarrow$ natural numbers and roots of unity produce complex numbers
- N -dimensional Hilbert space over \mathcal{F} : $\mathbb{N}^N \xrightarrow{\mathbb{N} \rightarrow \mathcal{F}} \mathcal{H}_N$

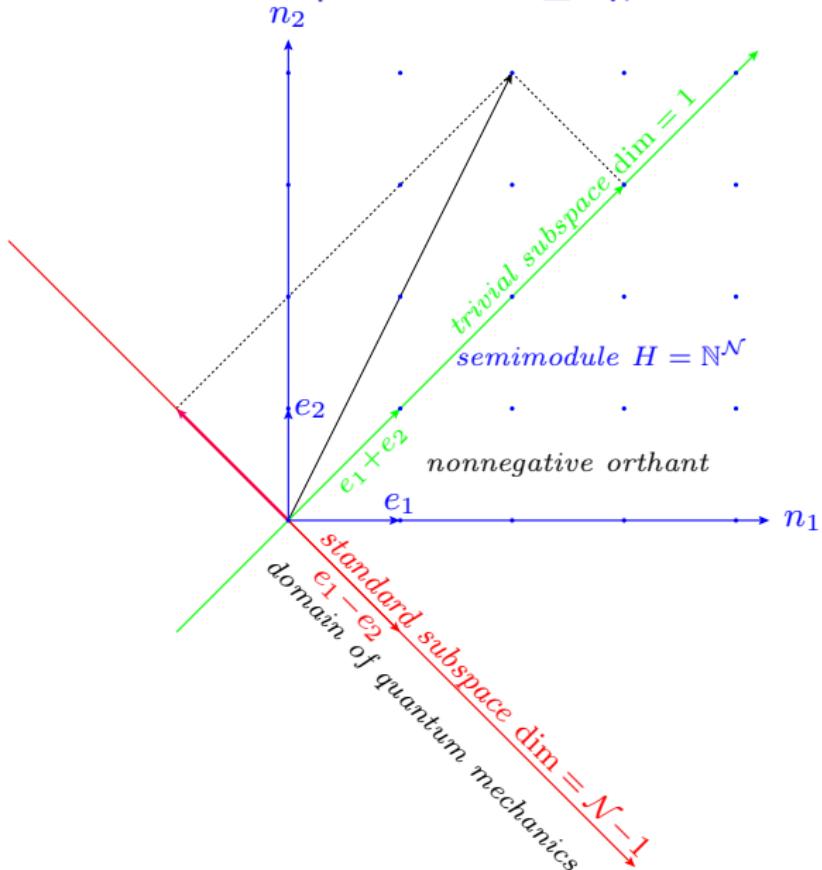
Any linear representation of G is
a subrepresentation of $\mathcal{P}(\mathcal{H}_N)$



Any quantum system can be
reproduced in a suitable
invariant subspace of \mathcal{H}_N

¹The term borrowed from G. 't Hooft, The Cellular Automaton Interpretation of Quantum Mechanics. *Fundamental Theories of Physics* 185. 296 p. Springer, 2016.

Basic invariant subspaces for $G \leq S_N$ derived from natural semimodule



Canonical bases

in trivial subspace

$$e_1 + e_2 + \cdots + e_N$$

all-ones vector

in standard subspace

$$e_1 - e_2$$

$$e_2 - e_3$$

⋮

$$e_{N-1} - e_N$$

Projector on standard subspace

$$P_{\text{std}} = \mathbb{1}_N - \frac{1}{N} \mathbb{J}_N$$

\mathbb{J}_N is all-ones matrix

Some arguments from physics and cosmology to support the approach

[★] Tom Banks, "Finite Deformations of Quantum Mechanics", arXiv:2001.07662[hep-th]

- Inclusion in the continuous unitary group of standard QM
 - ▶ for large enough \mathcal{N} (> 71 , T. Collins, 2007) most general finite subgroup of $SU(\mathcal{N} - 1)$ is a semidirect product
$$G = A \rtimes S_{\mathcal{N}}, \text{ where } A \text{ is an abelian group}$$
 - ▶ in [★] homocyclic group $A = \mathbb{Z}_k^{\mathcal{N}}$ is considered
remark: in this case $G = \mathbb{Z}_k^{\mathcal{N}} \rtimes S_{\mathcal{N}} \cong \mathbb{Z}_k \wr S_{\mathcal{N}}$ is a wreath product
- For fundamental (Planck) degrees of freedom [★] gives estimations
$$\mathcal{N} \sim \text{Exp}(10^{20})$$
 for 1 cm^3 of matter and $\mathcal{N} \sim \text{Exp}(10^{123})$ for the entire universe, whereas

"the entire history of the universe takes only 10^{61} Planck times",
thus

"the lifetime of any localized object, from the point of view of any static measuring apparatus, is indeed $\ll \mathcal{N}$."
- Final remark of [★]

"The work ... shows that the mathematical fact that ... naturally generates a set of truly quantum systems, which can encompass finite dimensional approximations to all known models of theoretical physics."

Ontic quantum states

- $(\mathcal{N} - 1)$ -dimensional standard Hilbert space $\mathcal{H}_{\mathcal{N}-1}$
- any pure quantum state $|\psi\rangle \in \mathcal{H}_{\mathcal{N}-1}$ is, with arbitrary precision, a normalized projection of some $|n\rangle \in \mathbb{N}^{\mathcal{N}}$
- normalized inner product in space $\mathcal{H}_{\mathcal{N}-1}$ in terms of natural vectors

$$\frac{\langle n | P_{\text{std}} | m \rangle}{\|P_{\text{std}}|n\rangle\| \cdot \|P_{\text{std}}|m\rangle\|} = S(n, m) = \frac{\mathcal{N}\langle n | m \rangle - \langle n \rangle \langle m \rangle}{\sqrt{\mathcal{N}\langle n | n \rangle - \langle n \rangle^2} \sqrt{\mathcal{N}\langle m | m \rangle - \langle m \rangle^2}},$$

where $\langle n \rangle = \sum_{k=1}^{\mathcal{N}} n_k$, $\langle n | m \rangle = \sum_{k=1}^{\mathcal{N}} n_k m_k$.

- ontic vectors are natural vectors with coordinates from $\{0, 1\}$ except for vectors $(0, 0, \dots, 0)^T$ and $(1, 1, \dots, 1)^T$
 - ▶ number of ontic vectors, $2^{\mathcal{N}} - 2$, grows rapidly with increasing \mathcal{N}
 - ▶ area of intersection of unit sphere with nonnegative orthant,

$$\frac{\mathcal{N}\pi^{\mathcal{N}/2}}{2^{\mathcal{N}}\Gamma(\mathcal{N}/2 + 1)} \sim \sqrt{\frac{\mathcal{N}}{\pi}} \left(\frac{e\pi}{2\mathcal{N}}\right)^{\mathcal{N}/2}, \text{ decreases rapidly with } \mathcal{N}$$

- ontic vector $|q\rangle$ can be written as bit string of length \mathcal{N}
- $|q\rangle$ describes partition of ontic set into two nontrivial subsets:

$$\Omega = \Omega_q \sqcup \Omega_{\sim q}$$

$\sim q$ is bitwise NOT for the bit string q , i.e. $\Omega_{\sim q} = \Omega \setminus \Omega_q$

- normalized inner product in standard Hilbert space $\mathcal{H}_{\mathcal{N}-1}$ for ontic vectors

$$S(q, r) = \frac{\mathcal{N}\langle q \& r \rangle - \langle q \rangle \langle r \rangle}{\sqrt{\langle q \rangle \langle \sim q \rangle \langle r \rangle \langle \sim r \rangle}}$$

$q \& r$ is bitwise AND

$\langle a \rangle$ is the Hamming weight — number of 1's in the bit string a

- symmetry between subset Ω_a and its complement $\Omega_{\sim a}$ together with obvious identities

$$\langle \sim a \rangle = \mathcal{N} - \langle a \rangle \quad \text{and} \quad \langle a \& b \rangle + \langle a \& \sim b \rangle = \langle a \rangle$$

imply

$$S(q, r) = -S(\sim q, r) = -S(q, \sim r) = S(\sim q, \sim r)$$

for Born's probabilities

$$\mathbf{B}(q, r) = \mathbf{B}(\sim q, r) = \mathbf{B}(q, \sim r) = \mathbf{B}(\sim q, \sim r)$$

Composite quantum systems

M -component quantum system

- Indexing set:

$$X \cong \{0, \dots, M-1\}$$

- Entire Hilbert space:

$$\tilde{\mathcal{H}} = \bigotimes_{x \in X} \mathcal{H}_x$$

- Homogeneous system: all spaces \mathcal{H}_x are isomorphic: $\mathcal{H}_x \cong \mathcal{H}$
 X can be viewed as points of geometric space
- Local Hilbert space: isomorphism class \mathcal{H}
 $\dim \mathcal{H} = n \implies \dim \tilde{\mathcal{H}} = n^M \equiv N$
- Orthonormal basis of local Hilbert space \mathcal{H} :

$$|0\rangle = \begin{pmatrix} 1 \\ 0 \\ \vdots \end{pmatrix}, |1\rangle = \begin{pmatrix} 0 \\ 1 \\ \vdots \end{pmatrix}, \dots, |n-1\rangle = \begin{pmatrix} 0 \\ \vdots \\ n-1 \end{pmatrix}$$

- Orthonormal basis of entire Hilbert space $\tilde{\mathcal{H}} = \bigotimes_{x \in X} \mathcal{H}_x$:

$$|\tilde{0}\rangle = \begin{pmatrix} 1 \\ 0 \\ \vdots \end{pmatrix}, |\tilde{1}\rangle = \begin{pmatrix} 0 \\ 1 \\ \vdots \end{pmatrix}, \dots, |\mathcal{N}-1\rangle = \begin{pmatrix} 0 \\ \vdots \\ \mathcal{N}-1 \end{pmatrix}$$

via local basis

$$|\tilde{i}\rangle = |i_0\rangle \otimes |i_1\rangle \otimes \cdots \otimes |i_{M-1}\rangle \equiv \bigotimes_{x \in X} |i_x\rangle$$

one-to-one correspondence:

$$\tilde{i} = i_0 n^{M-1} + \cdots + i_{M-1} \equiv \sum_{m=0}^{M-1} i_m n^{M-m-1}$$

“big-endian” positional representation of \tilde{i} in base n

Subsystem of composite system

Any **mixed quantum state** can be obtained from a **pure state** in larger Hilbert space²

Suppose **bit string** $q = b_0 b_1 \cdots b_{N-1}$ describes (pure) **ontic state** of entire system

- **density matrix** of entire system ontic state in standard space:

$$\rho_x = \frac{P_{\text{std}} |q\rangle \langle q| P_{\text{std}}}{\langle q | P_{\text{std}} | q \rangle} \rightarrow (\rho_x)_{ij} = \begin{cases} \frac{\langle q \rangle}{N \langle \sim q \rangle} & \text{for } b_i + b_j = 0 \\ -\frac{1}{N} & \text{for } b_i + b_j = 1 \\ \frac{\langle \sim q \rangle}{N \langle q \rangle} & \text{for } b_i + b_j = 2 \end{cases}$$

- **reduced density matrix**

$$\rho_A = \text{tr}_{X \setminus A} \rho_x$$

describes quantum behavior of subsystem indexed by $A \subset X$
generally reduced matrix describes a **mixed state**

²The metaphor “Church of the Larger Hilbert Space” (J. A. Smolin) expresses the belief that any mixed state actually comes from a more fundamental pure state of a larger system.

Entanglement

Separable states are states of a composite system that are tensor products of states of subsystems

Entangled states are states that are not separable

Illustrative example: all ontic vectors in $\tilde{\mathcal{H}}_4 = \mathcal{H}_2 \otimes \mathcal{H}_2$

Basis

$$|\tilde{0}\rangle = |0\rangle \otimes |0\rangle$$

$$|\tilde{1}\rangle = |0\rangle \otimes |1\rangle$$

$$|\tilde{2}\rangle = |1\rangle \otimes |0\rangle$$

$$|\tilde{3}\rangle = |1\rangle \otimes |1\rangle$$

Separable ontic states

$$|\tilde{0}\rangle = |0\rangle \otimes |0\rangle$$

$$|\tilde{1}\rangle = |0\rangle \otimes |1\rangle$$

$$|\tilde{2}\rangle = |1\rangle \otimes |0\rangle$$

$$|\tilde{3}\rangle = |1\rangle \otimes |1\rangle$$

$$|\tilde{0}\rangle + |\tilde{1}\rangle = |0\rangle \otimes (|0\rangle + |1\rangle)$$

$$|\tilde{0}\rangle + |\tilde{2}\rangle = (|0\rangle + |1\rangle) \otimes |0\rangle$$

$$|\tilde{1}\rangle + |\tilde{3}\rangle = (|0\rangle + |1\rangle) \otimes |1\rangle$$

$$|\tilde{2}\rangle + |\tilde{3}\rangle = |1\rangle \otimes (|0\rangle + |1\rangle)$$

Entangled ontic states

$$|\tilde{1}\rangle + |\tilde{2}\rangle = |0\rangle \otimes |1\rangle + |1\rangle \otimes |0\rangle$$

$$|\tilde{0}\rangle + |\tilde{3}\rangle = |0\rangle \otimes |0\rangle + |1\rangle \otimes |1\rangle$$

$$|\tilde{0}\rangle + |\tilde{1}\rangle + |\tilde{2}\rangle = |0\rangle \otimes |0\rangle + |0\rangle \otimes |1\rangle + |1\rangle \otimes |0\rangle$$

$$|\tilde{0}\rangle + |\tilde{1}\rangle + |\tilde{3}\rangle = |0\rangle \otimes |0\rangle + |0\rangle \otimes |1\rangle + |1\rangle \otimes |1\rangle$$

$$|\tilde{0}\rangle + |\tilde{2}\rangle + |\tilde{3}\rangle = |0\rangle \otimes |0\rangle + |1\rangle \otimes |0\rangle + |1\rangle \otimes |1\rangle$$

$$|\tilde{1}\rangle + |\tilde{2}\rangle + |\tilde{3}\rangle = |0\rangle \otimes |1\rangle + |1\rangle \otimes |0\rangle + |1\rangle \otimes |1\rangle$$

Geometry from entanglement

Main idea: determine distances between subsystems via quantum correlations — the greater the correlation, the less the distance

Some proponents

 *Van Raamsdonk M.*

Building up spacetime from quantum entanglement.
Gen. Relativ. Grav. **42**, 2323–2329, 2010.

 *Maldacena J., Susskind L.*

Cool horizons for entangled black holes.
Fortsch. Phys. **61** (9), 781–811, 2013.

 *Cao C., Carroll S.M., Michalakis S.*

Space from Hilbert space:
Recovering geometry from bulk entanglement.
Phys. Rev. D **95**, 024031, 2017.

Time as an entanglement phenomenon: Page-Wootters mechanism

Entropies

Entropy of probability distribution $P = (p_1, \dots, p_n)$ is real function $H(P)$ with some natural properties including

- ① either $H(PQ) = H(P) + H(Q|P)$ – chain rule of conditional entropy
single Shannon entropy

$$H_1(P) = - \sum_{i=1}^n p_i \log p_i$$

- ② or weaker $H(PQ) = H(P) + H(Q)$ – additivity on independent distributions
family of Rényi entropies

$$H_\alpha(P) = \frac{1}{1-\alpha} \log \sum_{i=1}^n p_i^\alpha, \quad \alpha \geq 0, \quad \alpha \neq 1$$

completion by limiting cases

- $\alpha \rightarrow 1$ $H_1(P)$ — Shannon entropy
- $\alpha \rightarrow \infty$ $H_\infty(P) = -\log \max_i p_i$ — min-entropy

special cases

- $H_0(P) = \log |\text{supp}(P)|$ — Hartley or max-entropy
- $H_2(P) = -\log \sum_{i=1}^n p_i^2$ — collision entropy

Monotonicity: $\alpha < \beta \implies H_\alpha \geq H_\beta$

Quantum entropies

Quantum version of completed Rényi family

$$S_\alpha(\rho) = \frac{1}{1-\alpha} \log \text{tr}(\rho^\alpha), \quad 0 \leq \alpha \leq \infty, \quad \rho \text{ is density matrix}$$

- for all α 
- additive on **separable states**: $S_\alpha(\rho_1 \otimes \rho_2) = S_\alpha(\rho_1) + S_\alpha(\rho_2)$
 - zero on **pure states**
 - maximum value = $\log \dim \mathcal{H}$ on **maximally mixed states**

Special cases

- $S_0(\rho) = \log \text{rank}(\rho)$ — quantum **max-entropy**
- $S_1(\rho) = -\text{tr}(\rho \log \rho)$ — **von Neumann entropy**
- $S_\infty(\rho) = -\log \|\rho\|$ — quantum **min-entropy**
- our choice $S_2(\rho) = -\log \text{tr}(\rho^2)$ — quantum **collision entropy**
 - — does not hold chain rule in contrast to von Neumann entropy
 - + easy to calculate: $S_2(\rho) = -\log \left(\sum_{i=1}^n \rho_{ii}^2 + 2 \sum_{i=1}^{n-1} \sum_{j=i+1}^n |\rho_{ij}|^2 \right)$
 - + log of Born's probability "system observes itself" $\text{tr}(\rho_O \rho_S)$, $\rho_O = \rho_S = \rho$
 - + more sensitive than von Neumann entropy as "entanglement witness"

Measures of entanglement between subsystems $A \subseteq X$ and $B \subseteq X$

There are plenty of them, in particular, quantum mutual information

$$I(A, B) = S_1(\rho_A) + S_1(\rho_B) - S_1(\rho_{A \cup B})$$

defined for the von Neumann entropy

We use

$$I(A, B) = S_2(\rho_A) + S_2(\rho_B) - S_2(\rho_{A \cup B})$$

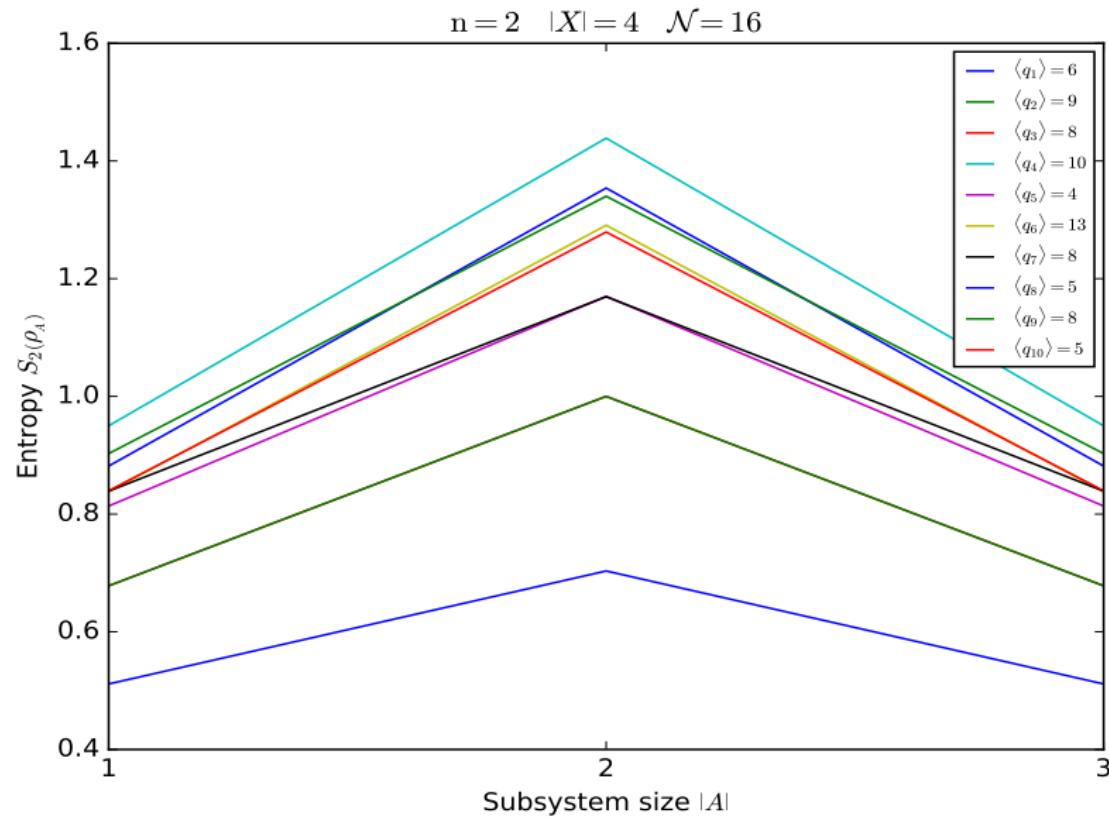
$I(A, B) = 0$ on separable and $I(A, B) \neq 0$ on entangled states

It is assumed: $\text{dist}(A, B) = \delta(I(A, B))$

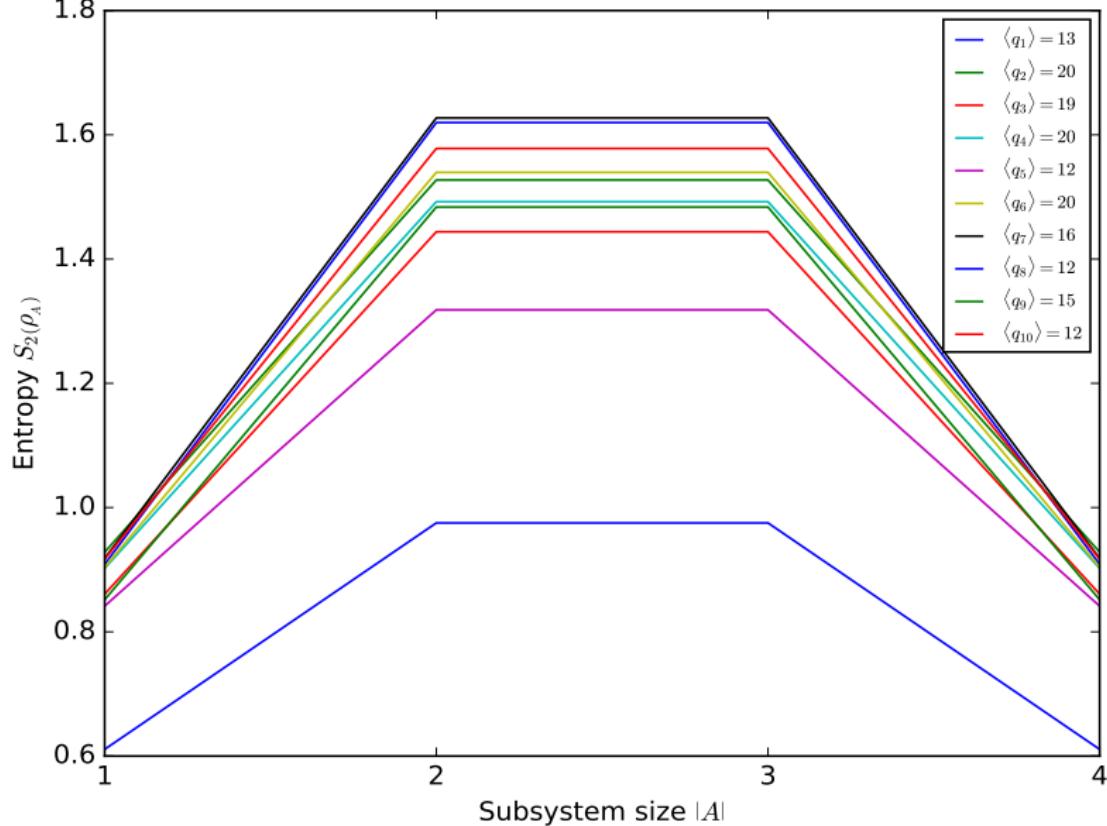
- δ is monotonically decreasing function
- $\delta(0) = \infty$
- $\delta(I(A, C)) \leq \delta(I(A, B)) + \delta(I(B, C))$ – subadditivity (triangle inequality)

Computer experiments: $S_2(\rho_A)$ vs $|A|$, $A \subset X$

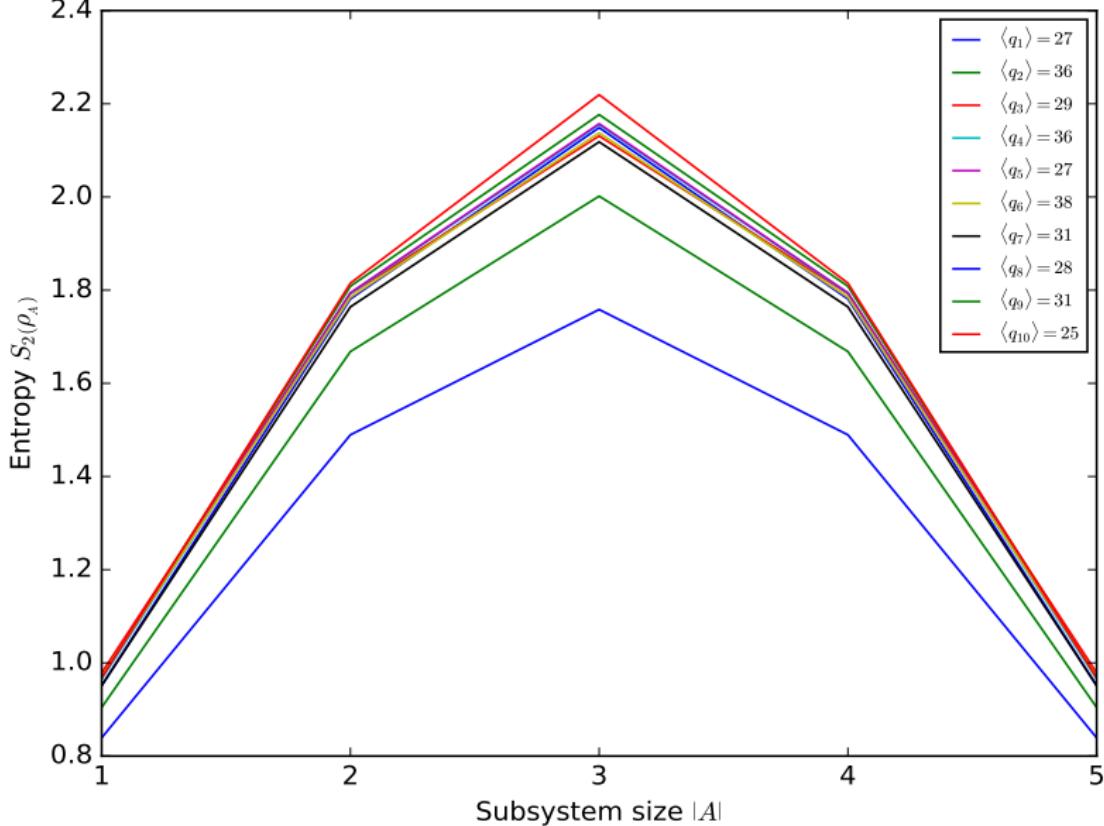
for maximally mixed state $S_2(\rho_A) = |A| \log n$



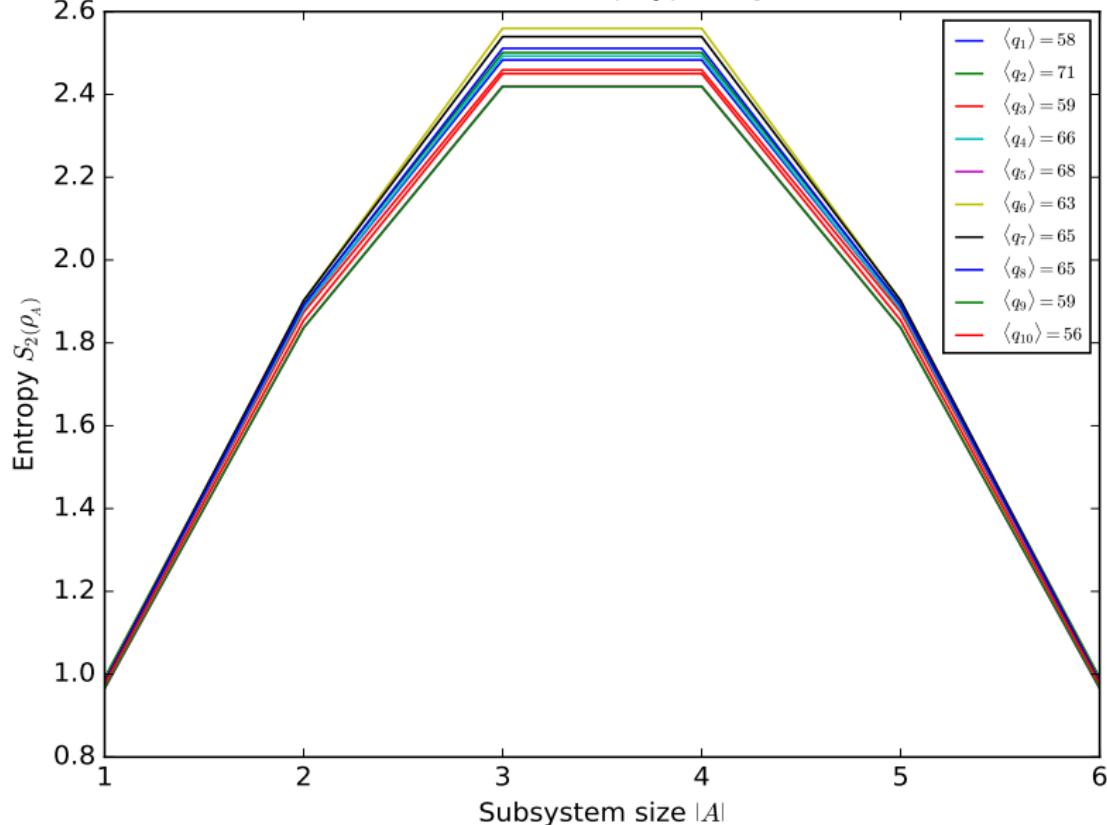
$n = 2$ $|X| = 5$ $\mathcal{N} = 32$



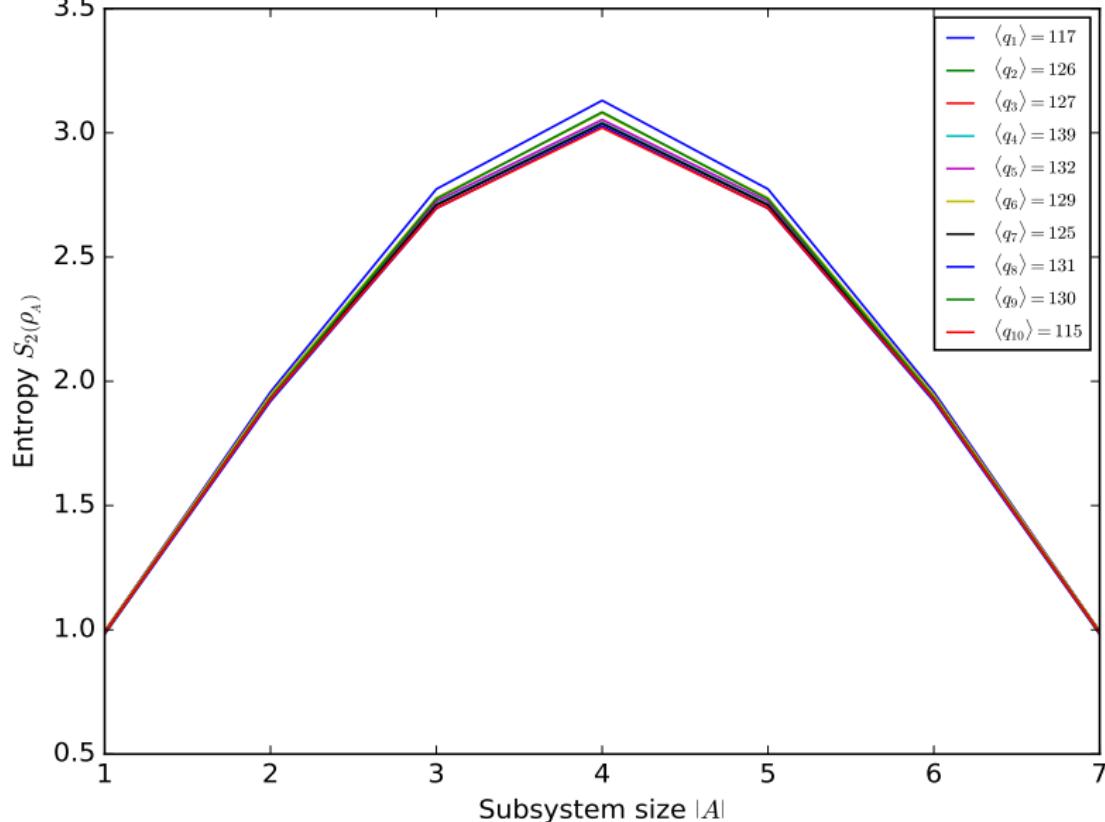
$n = 2$ $|X| = 6$ $\mathcal{N} = 64$



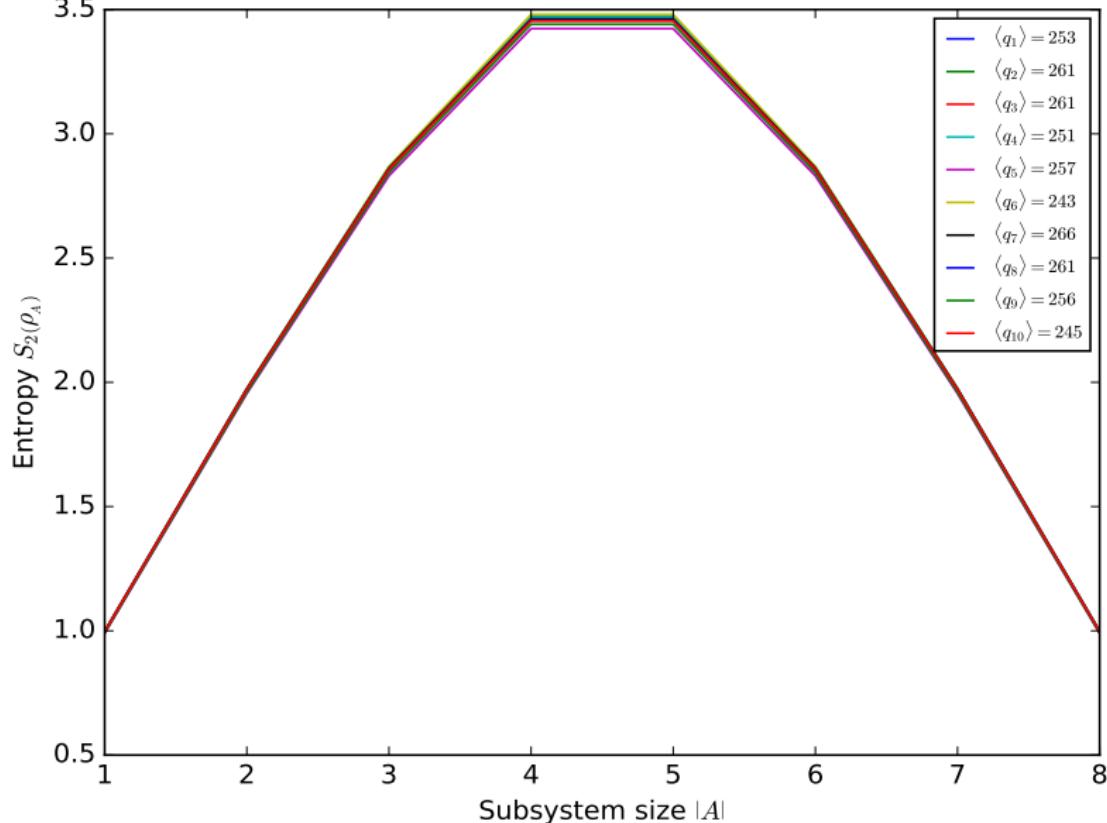
$n = 2$ $|X| = 7$ $\mathcal{N} = 128$



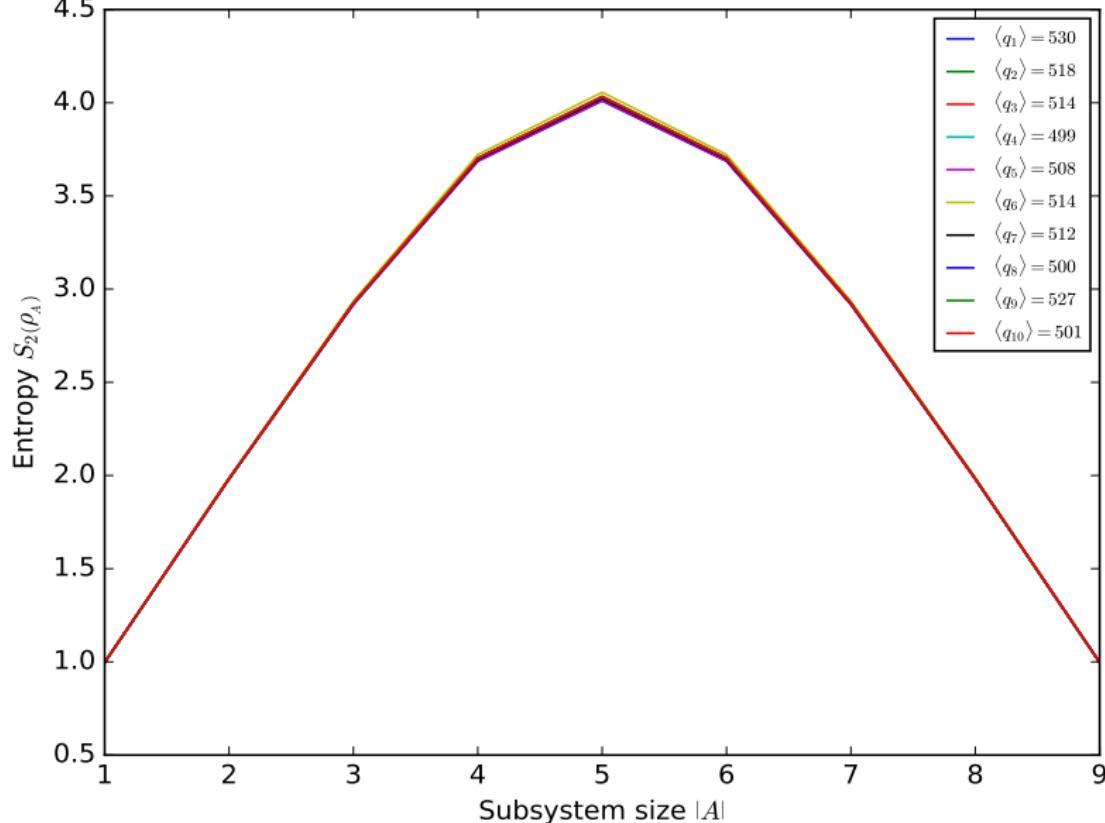
$n = 2$ $|X| = 8$ $\mathcal{N} = 256$



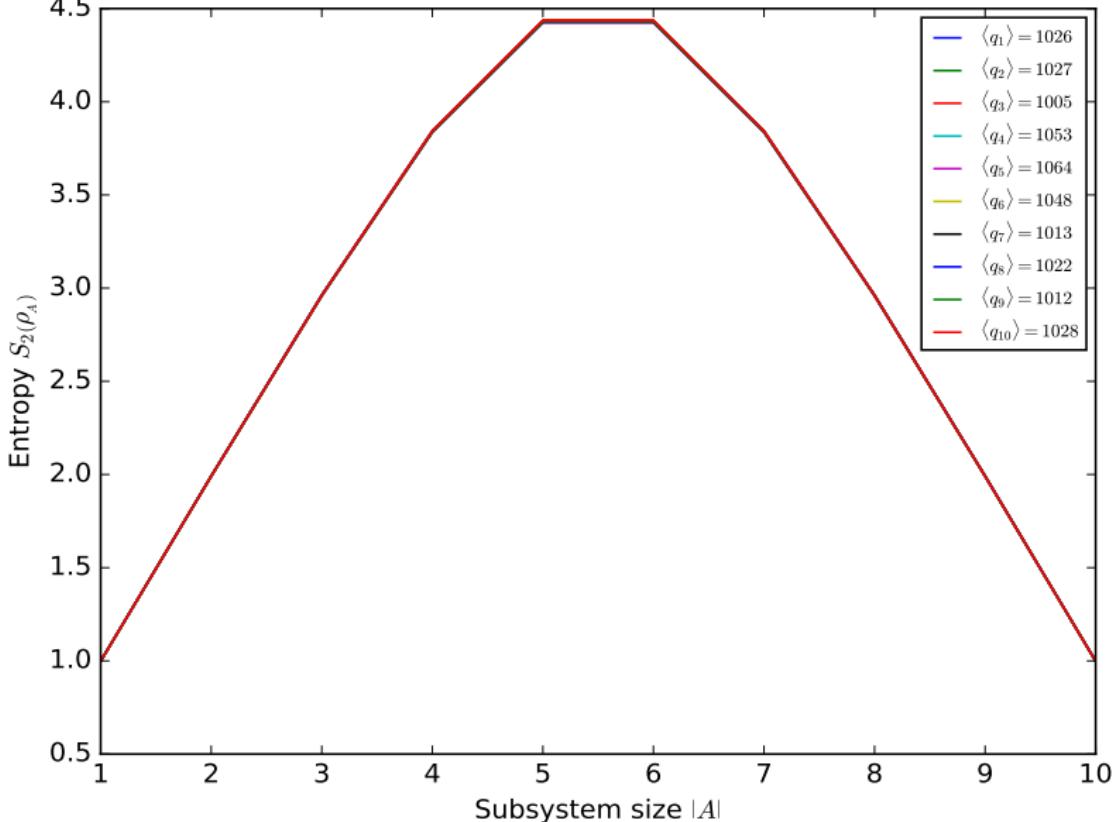
$n = 2$ $|X| = 9$ $\mathcal{N} = 512$

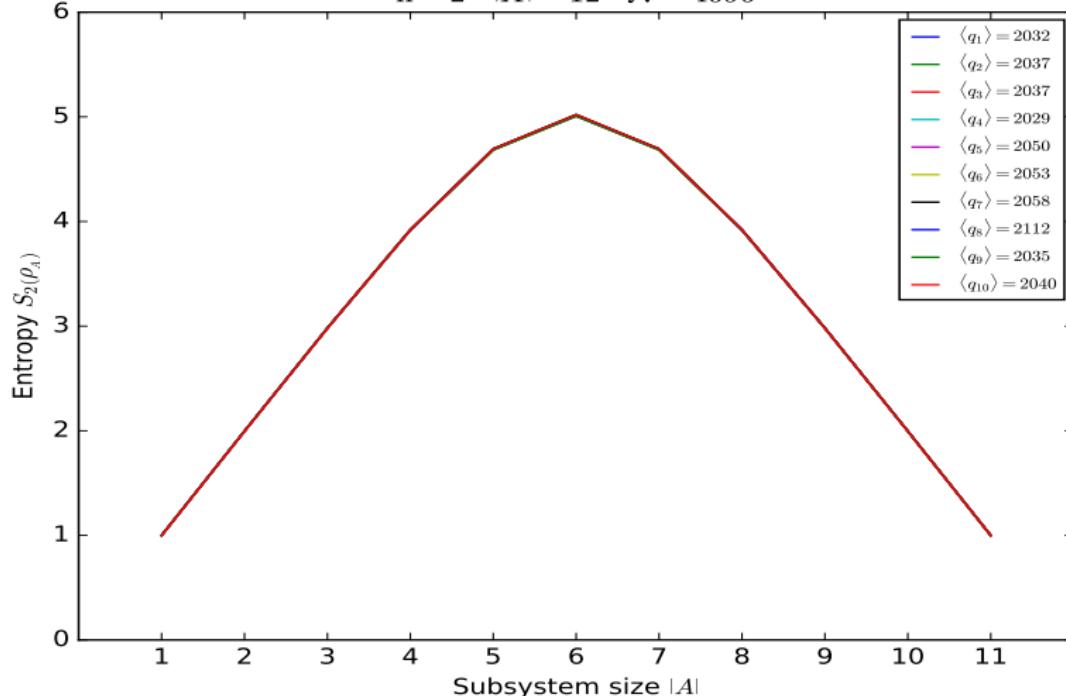


$n = 2$ $|X| = 10$ $\mathcal{N} = 1024$



$n = 2$ $|X| = 11$ $\mathcal{N} = 2048$



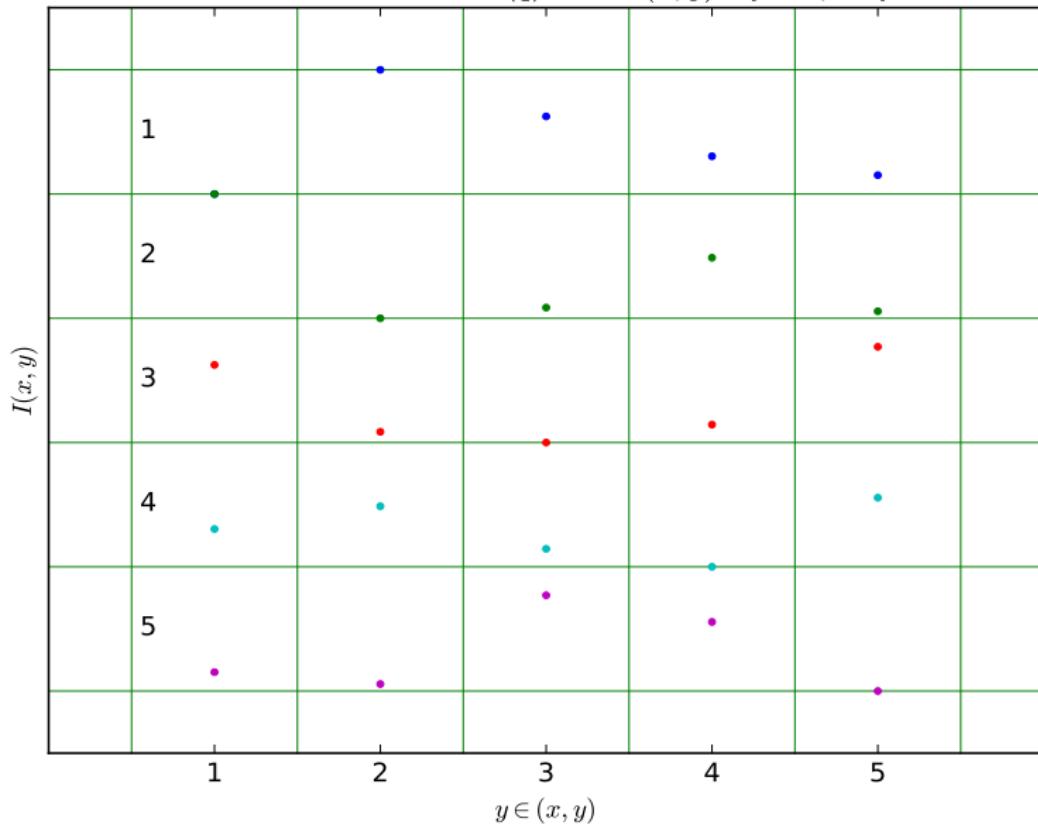


Observations:

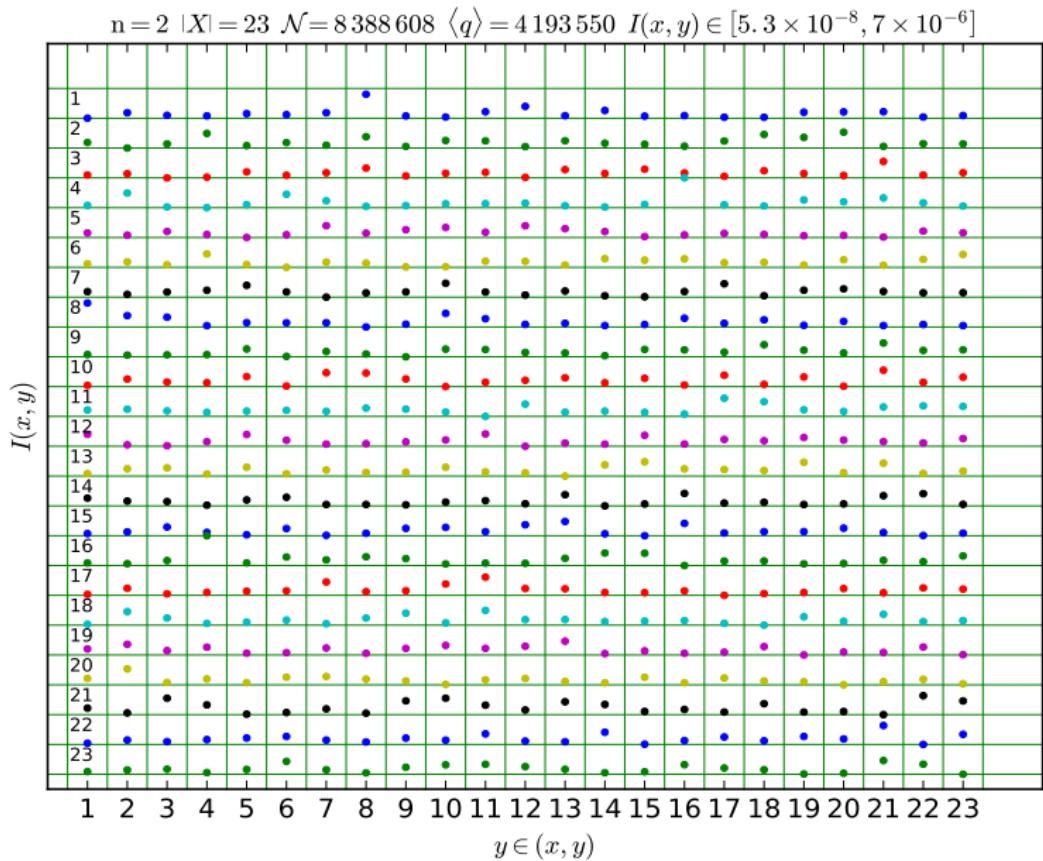
- Schmidt decomposition of pure state $\rho_x \implies$ symmetry $S_2(\rho_A) = S_2(\rho_{X \setminus A})$
- trend towards independence from quantum state with increasing $|X|$
- for $|A| < |X|/2$ close to maximally mixed state: $S_2(\rho_A) \approx |A| \log n$

Mutual information on edges of complete graph on X

$$n = 2 \ |X| = 5 \ \mathcal{N} = 32 \ \langle q \rangle = 13 \ I(x, y) \in [0.03, 0.5]$$



$I(x, y)$ on edges of X : large example



Increasing $n \rightarrow$ geometry of regular $(|X| - 1)$ -dimensional simplex

