The Newest Methods of Celestial Mechanics

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Abstract. Here for Hamiltonian systems we describe two of five methods of Celestial Mechanics. Namely: method of normal forms, allowing to study regular perturbations near a stationary solution, near a periodic solution, and method of truncated systems, found with a help of the Newton polyhedrons, allowing to study singular perturbations. Other three methods will be in the full presentation.

1. Normal forms

Here and below vectors in $\mathbb{R}^n$ or $\mathbb{C}^n$ are denoted by boldface font: $\mathbf{x} = (x_1, \ldots, x_n)$.

Let us consider the Hamiltonian system

$$\dot{\xi}_j = \frac{\partial \gamma}{\partial \eta_j}\,, \quad \dot{\eta}_j = -\frac{\partial \gamma}{\partial \xi_j}\,, \quad j = 1, \ldots, n \tag{1}$$

with n degrees of freedom in a vicinity of the stationary solution

$$\boldsymbol{\xi} = \boldsymbol{\eta} = 0. \tag{2}$$

If the Hamiltonian function $\gamma(\boldsymbol{\xi}, \boldsymbol{\eta})$ is analytic in the point (2), then it is expanded into the power series

$$\gamma(\boldsymbol{\xi}, \boldsymbol{\eta}) = \sum \gamma_{\mathbf{pq}} \boldsymbol{\xi}^{\mathbf{p}} \boldsymbol{\eta}^{\mathbf{q}}\,, \tag{3}$$

where $\mathbf{p}, \mathbf{q} \in \mathbb{Z}^n$, $\mathbf{p}, \mathbf{q} \geq 0$, $\boldsymbol{\xi}^{\mathbf{p}} = \xi_1^{p_1} \cdots \xi_n^{p_n}$. Here $\gamma_{\mathbf{pq}}$ are constant coefficients.

As the point (2) is stationary, than the expansion (3) begins from quadratic terms. They correspond to the linear part of the system (1). Eigenvalues of its matrix are decomposed in pairs:

$$\lambda_{j+n} = -\lambda_j\,, \quad j = 1, \ldots, n\,.$$

Let $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_n)$. The canonical changes of coordinates

$$\boldsymbol{\xi}, \boldsymbol{\eta} \longrightarrow \mathbf{x}, \mathbf{y} \tag{4}$$

preserve the Hamiltonian structure of the system.

Theorem 1 ([1, §12]). *There exists a formal canonical transformation* (4), *bringing Hamiltonian* (3) *to the normal form*

$$g(\mathbf{x},\mathbf{y}) = \sum g_{\mathbf{pq}} \mathbf{x}^{\mathbf{p}} \mathbf{y}^{\mathbf{q}} \tag{5}$$

contains only resonant terms with scalar product

$$\langle \mathbf{p}-\mathbf{q}, \boldsymbol{\lambda} \rangle = 0.$$

If $\boldsymbol{\lambda} \neq 0$, then the system corresponding to the normal form (5) is equivalent to a system with smaller number of degrees of freedom and with additional parameters. The normalizing transformation (4) conserves small parameters and linear automorphisms of the initial system (1)

$$\boldsymbol{\xi}, \boldsymbol{\eta} \longrightarrow \widetilde{\boldsymbol{\xi}}, \widetilde{\boldsymbol{\eta}}, , \quad t \to \widetilde{t}.$$

For the real initial system (1), the coefficients $g_{\mathbf{pq}}$ of the complex normal form (5) satisfy to special properties of reality and after a standard canonical linear change of coordinates $\mathbf{x}, \mathbf{y} \to \mathbf{X}, \mathbf{Y}$ Hamiltonian (5) transforms in a real one [2, Ch. I]. There are several methods of computation of coefficients $g_{\mathbf{pq}}$ of the normal form (5). The most simple method was described in the book [3]. Normal forms near a periodic solution, near an invariant torus and near family of them see in [2, Chs. II, VII, VIII], [4, Part II], [5], [6]. Normal form is useful in study stability, bifurcations and asymptotic behavior of solutions.

2. Truncated Hamiltonian functions

Let $\mathbf{x}$, $\mathbf{y}$ and $\boldsymbol{\mu} = (\mu_1, \ldots, \mu_s)$ be canonical variables and small parameters respectively. Let a Hamiltonian function be

$$h(\mathbf{x},\mathbf{y},\boldsymbol{\mu}) = \sum h_{\mathbf{pqr}} \mathbf{x}^{\mathbf{p}} \mathbf{y}^{\mathbf{q}} \boldsymbol{\mu}^{\mathbf{r}} \tag{6}$$

where $h_{\mathbf{pqr}}$ are constant coefficients and $\mathbf{r} \in \mathbb{Z}^s$, $\mathbf{r} \geq 0$. To each term of sum (6) we put in correspondence its vectorial power exponent $Q = (\mathbf{p}, \mathbf{q}, \mathbf{r}) \in \mathbb{R}^{2n+s}$. Set $\mathbf{S}$ of all points Q with $h_Q \neq 0$ in sum (6) is called as *support* $\mathbf{S} = \mathbf{S}(h)$ of the sum (6). The convex hull $\boldsymbol{\Gamma}(\mathbf{S}) = \boldsymbol{\Gamma}(h)$ of the support $\mathbf{S}$ is called as the *Newton polyhedron* of the sum (6). Its boundary $\partial\boldsymbol{\Gamma}(h)$ consists of vertices $\boldsymbol{\Gamma}_j^{(0)}$, edges $\boldsymbol{\Gamma}_j^{(1)}$ and faces $\boldsymbol{\Gamma}_j^{(d)}$ of dimensions d: $1 < d \leq 2n + s - 1$. Intersection $\mathbf{S} \bigcap \boldsymbol{\Gamma}_j^{(d)} = \mathbf{S}_j^{(d)}$ is the *boundary subset* of set $\mathbf{S}$. To each *generalized face* $\boldsymbol{\Gamma}_j^{(d)}$ (including vertices and edges) there correspond:

- *normal cone* $\mathbf{U}_j^{(d)}$ in space $\mathbb{R}_*^{2n+s}$, which is dual to space $\mathbb{R}^{2n+s}$;
- *truncated sum*

$$\hat{h}_j^{(d)} = \sum h_{\mathbf{pqr}} \mathbf{x}^{\mathbf{p}} \mathbf{y}^{\mathbf{q}} \boldsymbol{\mu}^{\mathbf{r}} \ \text{ over } \ Q = (\mathbf{p}, \mathbf{q}, \mathbf{r}) \in \mathbf{S}_j^{(d)}.$$

The truncated sum is the first approximation to the sum (6), when

$$(\log|x_j|, \log|y_j|, \log|\mu_k|) \to \infty, \quad j = 1, \dots, n, \quad k = 1, \dots, s,$$

near the normal cone $\mathbf{U}_j^{(d)}$.

So we can describe the approximate problems by truncated Hamiltonian functions. Example see below in Section 3.

3. Restricted 3-body problem

Let the two bodies $\mathbf{P}_1$ and $\mathbf{P}_2$ with masses $1-\mu$ and μ respectively turn in circular orbits around their common mass center with the period T. The plane circular restricted three-body problem consists in the study of the plane motion of the body $\mathbf{P}_3$ of infinitesimal mass under the influence of the Newton gravitation of bodies $\mathbf{P}_1$ and $\mathbf{P}_2$. In the rotating (synodical) standardized coordinate system the problem is described by the Hamiltonian system with two degrees of freedom and with one parameter μ [2]. The Hamiltonian function has the form

$$h \overset{\text{def}}{=} \frac{1}{2}\left(y_1^2 + y_2^2\right) + x_2 y_1 - x_1 y_2 - \frac{1-\mu}{\sqrt{x_1^2 + x_2^2}} - \frac{\mu}{\sqrt{(x_1-1)^2 + x_2^2}} + \mu x_1. \quad (7)$$

Here the body $\mathbf{P}_1 = \{X, Y : x_1 = x_2 = 0\}$ and the body $\mathbf{P}_2 = \{X, Y : x_1 = 1,\ x_2 = 0\}$, where $X = (x_1, x_2)$, $Y = (y_1, y_2)$. We consider the small values of the mass ratio $\mu \geq 0$. When $\mu = 0$ the problem turns into the two-body problem for $\mathbf{P}_1$ and $\mathbf{P}_3$. But here the points corresponding to collisions of the bodies $\mathbf{P}_2$ and $\mathbf{P}_3$ must be excluded from the phase space. The points of collisions split in parts solutions to the two-body problem for $\mathbf{P}_1$ and $\mathbf{P}_3$. For small $\mu > 0$ there is a singular perturbation of the case $\mu = 0$ near the body $\mathbf{P}_2$. In order to find all the first approximations to the restricted three-body problem, it is necessary to introduce the local coordinates near the body $\mathbf{P}_2$

$$\xi = x_1 - 1,\ \xi_2 = x_2,\ \eta_1 = y_1,\ \eta_2 = y_2 - 1$$

and to expand the Hamiltonian function in these coordinates. After the expansion of $1/\sqrt{(\xi_1+1)^2 + \xi_2^2}$ in the Maclaurin series, the Hamiltonian function (7) takes the form

$$\begin{aligned} h + \frac{3}{2} - 2\mu \overset{\text{def}}{=} & \frac{1}{2}(\eta_1^2 + \eta_2^2) + \xi_2\eta_1 - \xi_1\eta_2 - \xi_1^2 + \frac{1}{2}\xi_2^2 + \\ & + f\left(\xi_1, \xi_2^2\right) + \mu\left\{\xi_1^2 - \frac{1}{2}\xi_2^2 - \frac{1}{\sqrt{\xi_1^2 + \xi_2^2}} - f\left(\xi_1, \xi_2^2\right)\right\}, \end{aligned} \quad (8)$$

where f is the convergent power series, where the terms of order less then three are absent. Let for each term of sum (8) we put

$$p = \operatorname{ord}\xi_1 + \operatorname{ord}\xi_2, \quad q = \operatorname{ord}\eta_1 + \operatorname{ord}\eta_2, \quad r = \operatorname{ord}\mu.$$

Then support $\mathbf{S}$ of the expansion (8) consists of the points

$$(0,2,0), (1,1,0), (2,0,0), (k,0,0), (2,0,1), (-1,0,1), (k,0,1),$$

where $k = 3, 4, 5, \ldots$ The convex hull of the set $\mathbf{S}$ is the polyhedron $\mathbf{\Gamma} \subset \mathbb{R}^3$. The surface $\partial\mathbf{\Gamma}$ of the polyhedron $\mathbf{\Gamma}$ consists of faces $\mathbf{\Gamma}_j^{(2)}$, edges $\mathbf{\Gamma}_j^{(1)}$ and vertices $\mathbf{\Gamma}_j^{(0)}$. To each of the elements $\mathbf{\Gamma}_j^{(d)}$ there corresponds the truncated Hamiltonian $\hat{h}_j^{(d)}$, that is the sum of those terms of Series (8), the points $Q = (p, q, r)$ of which belong to $\mathbf{\Gamma}_j^{(d)}$. Fig. 1 shows the polyhedron $\mathbf{\Gamma}$, which is the semi-infinite trihedral prism with an oblique base. It has four faces and six edges. Let us consider them.

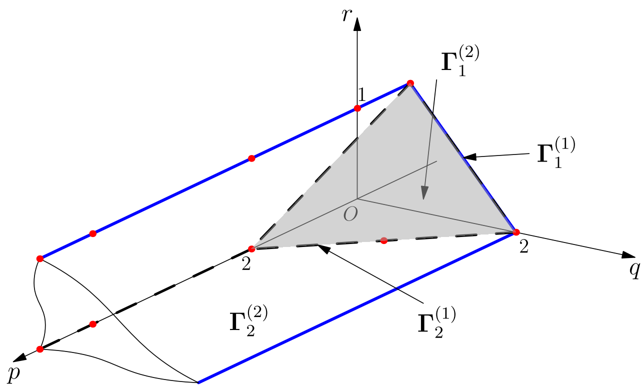

FIGURE 1. The polyhedron $\mathbf{\Gamma}$ for the Hamiltonian function (8) in coordinates p, q, r.

The face $\mathbf{\Gamma}_1^{(2)}$, which is the oblique base of the prism $\mathbf{\Gamma}$, contains vertices $(0, 2, 0)$, $(2, 0, 0)$, $(-1, 0, 1)$ and the point $(1, 1, 0) \in \mathbf{S}$ To the face there corresponds the truncated Hamiltonian function

$$\hat{h}_1^{(2)} = \frac{1}{2}\left(\eta_1^2 + \eta_2^2\right) + \xi_2\eta_1 - \xi_1\eta_2 - \xi_1^2 + \frac{1}{2}\xi_2^2 - \frac{\mu}{\sqrt{\xi_1^2 + \xi_2^2}}. \tag{9}$$

It describes the Hill problem [7], which is a non-integrable one. The canonical power transformation

$$\tilde{\xi}_i = \xi_i \mu^{-1/3}, \quad \tilde{\eta}_i = \eta_i \mu^{-1/3}, \quad i = 1, 2, \tag{10}$$

reduces the Hamiltonian (9) to the Hamiltonian of the form (9), where ξ_i, η_i, μ must be substituted by $\tilde{\xi}_i, \tilde{\eta}_1, 1$ respectively.

The face $\mathbf{\Gamma}_2^{(2)}$ contains points $(0, 2, 0)$, $(1, 1, 0)$, $(2, 0, 0)$ and $(k, 0, 0) \subset \mathbf{S}$. To the face there corresponds the truncated Hamiltonian function $\hat{h}_2^{(2)}$, which is obtained from the function h when $\mu = 0$. It describes the two-body problem for $\mathbf{P}_1$ and $\mathbf{P}_3$, which is an integrable one.

The edge $\mathbf{\Gamma}_1^{(1)}$ includes points $(0, 2, 0)$ and $(-1, 0, 1) \subset \mathbf{S}$. The corresponding truncated Hamiltonian function is

$$\hat{h}_1^{(1)} = \frac{1}{2}(\eta_1^2 + \eta_2^2) - \frac{\mu}{\sqrt{\xi_1^2 + \xi_2^2}}. \tag{11}$$

It describes the two-body problem for $\mathbf{P}_2$ and $\mathbf{P}_3$. The power transformation (10) transforms Hamiltonian (11) into the Hamiltonian function of the form (11), where ξ_i, η_i, μ must be substituted by $\tilde{\xi}_i, \tilde{\eta}_1, 1$ respectively.

The edge $\mathbf{\Gamma}_2^{(1)}$ includes points $(2,2,0), (1,1,0), (0,2,0) \subset \mathbf{S}$. To it there corresponds the truncated Hamiltonian function (9) with $\mu = 0$. It describes the intermediate problem (between the Hill problem and the two-body problem for $\mathbf{P}_1$ and $\mathbf{P}_3$), which is an integrable one. This first approximation was introduced by Hénon [8].

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References

[1] A. D. Bruno. Analytical form of differential equations (II). *Trans. Moscow Math. Soc.*, 26:199–239, 1972. = Trudy Moskov. Mat. Obsc. 25 (1971) 119-262 (in Russian).

[2] A. D. Bruno. *The Restricted 3–body Problem: Plane Periodic Orbits.* Walter de Gruyter, Berlin, 1994. = Nauka, Moscow, 1990. 296 p. (in Russian).

[3] V. F. Zhuravlev, A. G. Petrov, and M. M. Shunderyuk. *Selected Problems of Hamiltonian Mechanics.* LENAND, Moscow, 2015. (in Russian).

[4] A. D. Bruno. *Local Methods in Nonlinear Differential Equations.* Springer–Verlag, Berlin – Heidelberg – New York – London – Paris – Tokyo, 1989.

[5] A. D. Bruno. Normal form of a Hamiltonian system with a periodic perturbation. *Computational Mathematics and Mathematical Physics*, 60(1):36–52, 2020. = Zhurnal Vychislitelnoi Matematiki i Matematicheskoi Fiziki, 2020, V. 60(1): 39–56 (in Russian). https://doi.org/10.1134/S0965542520010066 doi:10.1134/S0965542520010066.

[6] A. D. Bruno. Normalization of the periodic Hamiltonian system. *Programming and Computer Software*, 46(2):76–83, 2020. = Programmirovanie, 2020, V. 46(2): 6–13 (in Russian). https://doi.org/10.31857/S0132347420020053 doi:10.31857/S0132347420020053.

[7] G. W. Hill. Researches in the lunar theory. *Amer. J. Math.*, 1:5–26, 129–147, 245–260, 1878.

[8] M. Hénon. Numerical exploration of the restricted problem. V. Hill's case: periodic orbits and their stability. *Astron. & Astrophys.*, 1:223–238, 1969.

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