Subexponential–time computation of isolated primary components of a polynomial ideal

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Let k be a field of arbitrary characteristic with an algebraic closure $\overline{k}$. Let H be a primitive subfield of the field k and $H(t_1, \ldots, t_l)$ be the field of rational functions over H in algebraically independent variables $t_1, \ldots, t_l$ over H. We assume that the field k is a finite separable extention $H(t_1, \ldots, t_l)$ given by its primitive element θ over $H(t_1, \ldots, t_l)$ (a minimal polynomial $\Phi \in H(t_1, \ldots, t_l)[Z]$ of the element θ is also given).

Let $f_1, \ldots, f_m \in k[X_1, \ldots, X_n]$ be polynomials of degree at most d where $d \geqslant 2$ is an integer. Denote by $I \subset \overline{k}[X_1, \ldots, X_n] = A$ the polynomial ideal generated by the polynomials $f_1, \ldots, f_m$. We suggest a simple algorithm for computing all the isolated primary components of the ideal I. At the output of his algorithm they are given up to embedded components.

More precisely, let $\mathfrak{p}$ be an arbitrary isolated associated prime ideal of the ideal I and $I_\mathfrak{p}$ be the $\mathfrak{p}$-primary component of the ideal I. Then for each $\mathfrak{p}$ we construct the field of fractions $K_\mathfrak{p}$ of the ring $A/\mathfrak{p}$, and the following objects.

1) A polynomial ideal $I'_\mathfrak{p} \subset \overline{k}[X_1, \ldots, X_n]$ such that $I_\mathfrak{p}$ is a unique isolated primary component of $I'_\mathfrak{p}$. The ideal $I'_\mathfrak{p}$ is given by its system of generators.
2) A finite dimensional $K_\mathfrak{p}$-algebra $K_\mathfrak{p} \otimes_A (A/I_\mathfrak{p})$. This algebra is given by its basis over $K_\mathfrak{p}$ and the multiplication table. Hence $I_\mathfrak{p}$ coincides with the kernel of the natural homomorphism $A \to K_\mathfrak{p} \otimes_A (A/I_\mathfrak{p})$.

Denote by $V = \mathcal{Z}(f_1, \ldots, f_m)$ the algebraic variety of all common zeroes of the polynomials $f_1, \ldots, f_m$ in $\mathbb{A}^n(\overline{k})$. Notice that the homomorphism $A/I \to K_\mathfrak{p} \otimes_A (A/I_\mathfrak{p})$ for a primary ideal $I_\mathfrak{p}$ is an analog of the generic point $\overline{k}[V] \to K_\mathfrak{p}$ of the irreducible component $\mathcal{Z}(\mathfrak{p})$ of the algebraic variety V.

The complexity of this algorithm is polynomial in d^{n^2} and the size of the input data. It seems that so far there has not been explicit estimates for the complexity of algorithms for this problem in the considered general situation.

Notice that that the varieties $V(\mathfrak{p})$ might be of distinct dimensions. To substantiate this algorithm we use a non–trivial estimation for the degrees of primary components of the ideal $\sum_{1\leqslant i\leqslant m} Af_i$ in the case of homogeneous polynomials f_i obtained by the author earlier [1].

References

[1] A.L. Chistov, *Inequalities for Hilbert functions and primary decompositions*, Algebra i Analiz v. 19 (2007) No. 6, p. 143–172 (in Russian) [English transl.: St. Petersburg Math. J., v. 19 (2008) No. 6, p. 975–994].

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