Surface electromagnetic waves

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Introduction

In the report we discuss surface electromagnetic waves propagating along the boundary of isotropic and anisotropic media. In the 1980s, surface waves were discovered that propagate along the interface between two dielectrics without loss. We show how these waves can be investigated in CAS Sage.

Permittivity

The first analytical expressions were obtained manually for solutions that are waves propagating along the interface of an anisotropic medium with permittivity

$$\varepsilon = \mathrm{diag}(\varepsilon_o, \varepsilon_o, \varepsilon_e).$$

and isotropic medium with constant permittivity ε. For definiteness, let the plane $x = 0$ serve as the interface.

Field

Let there be a substance (field) with a dielectric permittivity

$$\varepsilon = \mathrm{diag}(\varepsilon_o, \varepsilon_o, \varepsilon_e).$$

Looking for a field of the type

$$\vec{E}(x, y, z, t) = \vec{E} e^{i\vec{k}\cdot\vec{r} - i\omega t}, \quad \omega/c = k_0.$$

Here $\vec{E}$ vector intensity of electric field, ω is the circular frequency of the wave, k_0 is the wave number.

Maxwell's equations

Maxwell's equations are reduced to an algebraic equation

$$\vec{k} \times [\vec{k} \times \vec{E}] + k_0^2 \vec{D} = 0.$$

Equating its determinant to zero, we see that it has a nontrivial solution in two cases: when k_x is the root of the equation

$$k_x^2 = \varepsilon_o k_0^2 - k_y^2 - k_z^2, \tag{1}$$

which is further denoted as k_{xo}, and when k_x is the root of the equation

$$k_x^2 = \varepsilon_e k_0^2 - k_y^2 - \frac{\varepsilon_e}{\varepsilon_o} k_z^2. \tag{2}$$

which is further denoted as k_{xe}.

Results for the roots k_{xo}, k_{xe}

In the former case, the solution is given by the formula

$$\vec{E}_o = \begin{pmatrix} k_y \\ -k_{xo} \\ 0 \end{pmatrix}, \quad \vec{H}_o = \frac{1}{k_0}\vec{k} \times \vec{E}_o = \frac{1}{k_0}\begin{pmatrix} k_{xo}k_z \\ k_y k_z \\ k_z^2 - \varepsilon_o k_0^2 \end{pmatrix}$$

In the second case, the solution is given by the formula

$$\vec{E}_e = -\begin{pmatrix} k_{xe}k_z \\ k_y k_z \\ k_z^2 - \varepsilon_o k_0^2 \end{pmatrix}, \quad \vec{H}_e = \frac{1}{k_0}\vec{k} \times \vec{E}_e = \varepsilon_o k_0 \begin{pmatrix} k_y \\ -k_{xe} \\ 0 \end{pmatrix}$$

The isotropic field

For the isotropic medium $(x > 0)$ the field is described by formulas

$$\vec{E} = \left(b_o \vec{E}'_o + b_e \vec{E}'_e\right) e^{-px} e^{ik_y y + ik_z z - i\omega t},$$
$$\vec{H} = \left(b_o \vec{H}'_o + b_e \vec{H}'_e\right) e^{-px} e^{ik_y y + ik_z z - i\omega t},$$

but now the constant p, which characterizes the decrease in the field in the isotropic medium, turns out :

$$p^2 = k_y^2 + k_z^2 - \varepsilon k_0^2.$$

The anisotropic field

The field in the anisotropic medium $(x < 0)$ is sought in the form

$$\vec{E} = \left(a_o \vec{E}_o e^{p_o x} + a_e \vec{E}_e e^{p_e x}\right) e^{ik_y y + ik_z z - i\omega t},$$
$$\vec{H} = \left(a_o \vec{H}_o e^{p_o x} + a_e \vec{H}_e e^{p_e x}\right) e^{ik_y y + ik_z z - i\omega t}.$$

Here
$\vec{k}_\perp = (0, k_y, k_z)$ is its wave vector, a_o, a_e is the amplitude of two partial waves, and positive numbers p_o, p_e characterize the rate of wave decay in the anisotropic medium. Maxwell's equations give

$$p_o^2 = k_y^2 + k_z^2 - \varepsilon_o k_0^2$$
$$p_e^2 = k_y^2 + \frac{\varepsilon_e}{\varepsilon_o} k_z^2 - \varepsilon_e k_0^2$$

.

Conditions for matching electromagnetic fields

The conditions for matching electromagnetic fields at the interface lead to a system of homogeneous linear equations for the amplitudes a_o, a_e, b_o, b_e. The condition of zero determinant of this system gives the equation

$$\begin{aligned}&\left((k_z^2 - \varepsilon k_0^2)p_o + (k_z^2 - \varepsilon_o k_0^2)p\right)\\&\left((k_z^2 - \varepsilon k_0^2)\varepsilon_o p_e + (k_z^2 - \varepsilon_o k_0^2)\varepsilon p\right)\\&= (\varepsilon_o - \varepsilon)^2 k_y^2 k_z^2 k_0^2\end{aligned} \tag{3}$$

The field satisfying Maxwell's equations

If in the domain of variation of five variables $k_y k_z p_o p_e p$, in the region specified by the inequalities

$$p_o > 0, \quad p_e > 0, \quad p > 0,$$

the system of algebraic equations

$$\begin{cases} p_o^2 = k_y^2 + k_z^2 - \varepsilon_o k_0^2 \\ p_e^2 = k_y^2 + \dfrac{\varepsilon_e}{\varepsilon_o} k_z^2 - \varepsilon_e k_0^2 \\ p^2 = k_y^2 + k_z^2 - \varepsilon k_0^2 \\ \big((k_z^2 - \varepsilon k_0^2) p_o + (k_z^2 - \varepsilon_o k_0^2) p\big) \cdot \\ \big((k_z^2 - \varepsilon k_0^2) \varepsilon_o p_e + (k_z^2 - \varepsilon_o k_0^2) \varepsilon p\big) = (\varepsilon_o - \varepsilon)^2 k_y^2 k_z^2 k_0^2 \end{cases}$$

has a solution, then this solution corresponds to a field satisfying Maxwell's equations, matching conditions at the interface.

The system of equations

It is possible to eliminate k_0 from this system by assuming

$$p = k_0 q, \quad p_o = k_0 q_o, \quad p_e = k_0 q_e$$

and

$$k_y = k_0 \beta, \quad k_z = k_0 \gamma.$$

Then the system of equations is written in the form

$$\begin{cases} q_o^2 = \beta^2 + \gamma^2 - \varepsilon_o \\ q_e^2 = \beta^2 + \dfrac{\varepsilon_e}{\varepsilon_o}\gamma^2 - \varepsilon_e \\ q^2 = \beta^2 + \gamma^2 - \varepsilon \\ \big((\gamma^2 - \varepsilon)q_o + (\gamma^2 - \varepsilon_o)q\big) \\ \big((\gamma^2 - \varepsilon)\varepsilon_o q_e + (\gamma^2 - \varepsilon_o)\varepsilon q\big) = \\ = (\varepsilon_o - \varepsilon)^2 \beta^2 \gamma^2. \end{cases} \tag{4}$$

The system of equations

This system (4) defines a curve in the space $\beta\gamma q q_o q_e$ and we are interested to know whether this curve falls within the region

$$q > 0, \quad q_o > 0, \quad q_e > 0.$$

The following observation allowed us to move forward: this curve can be described relatively simply if we consider its projection not on the $\beta\gamma$ plane, which we tried to do first of all, but on the $q_o q_e$ plane.
To study this system, we will exclude the variables β, γ, q. After exclusion, an equation of the form is obtained

$$F(q_o, q_e) = 0,$$

whose coefficients depend only on the permittivity.

Theory of elimination ideal

The theory of elimination ideal gives the equation

$$F(q_o, q_e) = 0$$

as a necessary and sufficient condition for the existence of a solution of (4), but this solution can be complex and infinitely large. In this case, it is possible to express the solution through the parameter t explicitly and explicitly investigate the fall of this solution into the real domain with the conditions

$$q_o > 0, \quad q_e > 0, \quad q > 0.$$

The first factor

Consider the multipliers separately.
System (4) has a solution

$$q_e = q_o.$$

Then the difference of the first two equations of the system (4) gives

$$\gamma^2 = \varepsilon_o.$$

This kind of solution does exist, but

$$k_z^2 - \varepsilon_o k_0^2 = 0,$$

therefore, $b_o = b_e = 0$ (the fields in the isotope are zero), and $\vec{E}_o$ and $\vec{E}_e$ become linearly dependent. Therefore, even for non-zero a_o, a_e the field in anisotropy can be zero.

The second factor

The second factor gives

$$(q_e^2 - q_o^2)\varepsilon_o = (\varepsilon - \varepsilon_o)(\varepsilon_e - \varepsilon_o).$$

Difference of the first three equations of the system (4) gives

$$(q_e^2 - q_o^2)\varepsilon_o = (\varepsilon_e - \varepsilon_o)(\gamma^2 - \varepsilon_o).$$

From here

$$\gamma^2 = \varepsilon.$$

This is the second trivial case: now $a_o = a_e = 0$ and the field has no anisotropy. Discarding trivial solutions, we see that the system (4) has a solution if and only if the third multiplier vanishes.

The third factor

The expression for this factor three is large, so let's note its structure for now

$$F_4(p_o, p_e) + F_2(p_o, p_e) = 0,$$

F_4, F_2 – homogeneous functions of the 4th and 2nd orders. Assuming

$$p_e = tp_o,$$

we rewrite this equation as

$$q_o^2 F_4(1,t) + F_2(1,t) = 0.$$

From here

$$q_o = \sqrt{\frac{-F_2(1,t)}{F_4(1,t)}}, \quad q_e = t\sqrt{\frac{-F_2(1,t)}{F_4(1,t)}},$$

where t can take any values.

The t parameter

Note that this area includes solutions that correspond to positive values of the parameter $t = q_e/q_o$.
For any $t > 0$ the equation

$$q_o^2 F_4(1,t) + F_2(1,t) = 0.$$

has two solutions. If

$$\frac{-F_2(1,t)}{F_4(1,t)} > 0,$$

then one of them,

$$q_o = \sqrt{\frac{-F_2(1,t)}{F_4(1,t)}}$$

positive. For such values of t, q_o the system (4) has a solution.

The value of β and γ

The values β^2, γ^2 can be restored by solving the system

$$\begin{cases} \beta^2 + \gamma^2 = q_o^2 + \varepsilon_o \\ \varepsilon_o \beta^2 + \varepsilon_e \gamma^2 = \varepsilon_o q_e^2 + \varepsilon_o \varepsilon_e \end{cases}$$

From here

$$\beta = \pm\sqrt{\frac{(\varepsilon_e - \varepsilon_o t^2) q_o^2}{\varepsilon_e - \varepsilon_o}} \qquad (5)$$

and

$$\gamma = \pm\sqrt{\frac{\varepsilon_o(\varepsilon_e - \varepsilon_o + (t^2 - 1) q_o^2)}{\varepsilon_e - \varepsilon_o}} \qquad (6)$$

Since the system (4) includes only squares β, γ, the solution will be obtained for any choice of signs before the radicals.

Value q

The value q can take at least one of two values:

$$q = \pm\sqrt{q_o^2 + \varepsilon_o - \varepsilon}.$$

For this value to be real, the inequality has to be fulfilled

$$q_o^2 > \varepsilon - \varepsilon_o.$$

In the last equation of the (4) system, the value q is included in the first power, so we have no right to choose the sign before the root at will. To check the correctness of the choice of the sign — we need to draw a graph of the left side of the last of the equations (4) for

$$q = \sqrt{q_o^2 + \varepsilon_o - \varepsilon}.$$

The case of Dyakonov

Dyakonov's case: $\varepsilon_o < \varepsilon < \varepsilon_e$. In this case:

$$F_2(1,t) < 0,$$

and the sign of $-F_4(1,t)$ is determined by the sign of the factor

$$\varepsilon_o t^2 + (\varepsilon - \varepsilon_o)t + \varepsilon - 2\varepsilon_e.$$

Therefore, $-F_2/F_4$ will be positive only between the roots of the equation

$$\varepsilon_o t^2 + (\varepsilon - \varepsilon_o)t + \varepsilon - 2\varepsilon_e = 0.$$

Since $\varepsilon - 2\varepsilon_e < 0$, these roots have different signs, so we should limit the change in t to zero and the positive root of this equation, denoting it as t_1.

Boundaries of q and t changes

Positivity q gives

$$q_o^2 > \varepsilon - \varepsilon_o.$$

Reality β gives

$$t < \sqrt{\frac{\varepsilon_e}{\varepsilon_o}},$$

and reality γ gives

$$\varepsilon_e - \varepsilon_o + (t^2 - 1)q_o^2 > 0,$$

We got the following bounds for changing t

$$0 < t < \min(t_1, \sqrt{\frac{\varepsilon_e}{\varepsilon_o}}),$$

and for q_o

$$\varepsilon - \varepsilon_o < q_o^2 < \frac{\varepsilon_e - \varepsilon_o}{1 - t^2)}.$$

Graph q_o

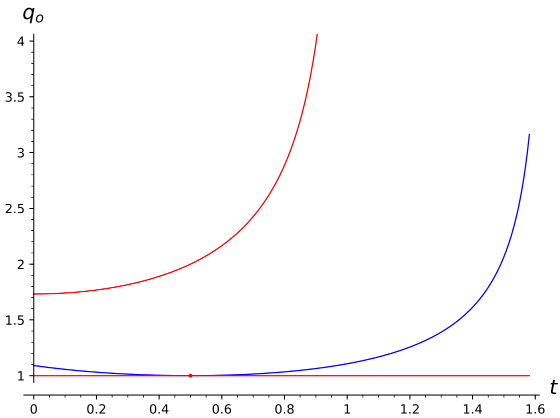

Figure: Graph q_o (blue line), red lines indicate the boundaries of the range t, q_o

Graph u

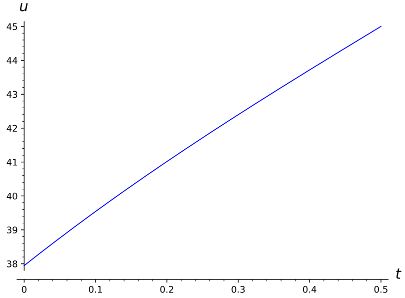

Figure: Graph u the angle value is specified in degrees

The graph of the dependence of the angle u on t, given by the formula

$$\frac{\gamma}{\beta} = \frac{k_z}{k_y} = \tan u \qquad (7)$$

Conclusion

- We considered surface waves at the boundary of two anisotropy and isotope fields.
- We obtained an algebraic curve in a five-dimensional space.
- Nontrivial solutions for the existence of surface waves are found
- The projection on the $q_0 q_e$ plane is used

Thanks for your attention