On invariant coordinate subspaces of normal form of ODE system

Alexander Batkhin

Abstract. A system of ODEs with non-degenerate linear part near its stationary point are considered in two cases: in general case and in Hamiltonian case. Solution of the problem of existence of an invariant coordinate subspace in the coordinates of normal form is proposed as a resonance relation between system's eigenvalues. Algorithms of computer algebra and q-subdiscriminant technique are used for finding such resonance relations.

Introduction

An approach of Poincaré for investigation of systems of nonlinear ordinary differential equations was based on the maximal simplification of the right-hand sides of these equations by invertible transformations. This approach led to the theory of normal forms (NF) of the general system and in particularly of the Hamiltonian ones and was developed in works of G.D. Birkhoff, T.M. Cherry, F.G. Gustavson, C.L. Siegel, J. Moser, A.D. Bruno (see [1]).

The goal of the presented work is to investigate invariant coordinates subspaces of NF of a real Hamiltonian system with non-degenerated linear part. The existence of invariant subspace can reduce the phase flow on the space of less dimension and in some cases can give information about periodic solution of the whole system.

1. Invariant subspaces of normal form of ODE

Consider an analytical system of ODE

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) \tag{1}$$

near its stationary point $\mathbf{x} = 0$. Let the linear part

$$\dot{\mathbf{x}} = A\mathbf{x}, \quad A = \partial\mathbf{f}/\partial\mathbf{x}|_{\mathbf{x}=0}\,, \tag{2}$$

of the system (1) be non-degenerated. Let matrix A has n eigenvalues at least one of which is non-zero $\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_n)$.

There exists [2] a formal invertible transformation $\mathbf{g} : \mathbf{x} \to \mathbf{y}$, $\mathbf{x} = \mathbf{g}(\mathbf{y})$, represented in the form of power series, which reduces the initial system (1) into its *normal form*

Definition 1. *Normal form* (NF) of the initial system (1) is a system of the form

$$\dot{y}_j = y_j h_j(\mathbf{y}), \quad j = 1, \dots, n, \tag{3}$$

right-hand sides $y_j h_j(\mathbf{y})$ of which are power series

$$y_j h_j(\mathbf{y}) = y_j \sum_{\mathbf{q}} h_{j\mathbf{q}} \mathbf{y}^{\mathbf{q}}, \quad h_{j0} = \lambda_j, \quad j = 1, \dots, n, \tag{4}$$

containing only resonant terms with

$$\langle \mathbf{q}, \boldsymbol{\lambda} \rangle = 0. \tag{5}$$

Here $h_{j\mathbf{q}}$ are constant coefficients and in $y_j h_j(\mathbf{y})$ coordinate $q_j \geq -1$, but others $q_k \geq 0$.

Coordinate subspace Let $I = \{i_1, \dots, i_k\}$ be a set of increasing indices $1 \leq i_1$, $i_k \leq n$, $k \leq n$. By K_I we denote the *coordinate subspace* $K_I = \{\mathbf{y} : y_j = 0 \text{ for all } j \notin I\}$. All non-zero coordinates y_j, $j \in I$, of the subspace K_I we call *internal coordinates* and denote them shortly by $\mathbf{y}_I$, others we call *external coordinates.* The eigenvalues λ_j, $j \in I$, corresponding to the internal coordinates $\mathbf{y}_I$ we call *internal eigenvalues* and denote them by $\boldsymbol{\lambda}_I$. Others λ_j, $j \notin I$, are called *external eigenvalues.*

Problem. Which subspaces K_I are invariant in the normal form (3), (4), (5)?

Theorem 1. *The coordinate subspace K_I of dimension k is invariant in the normal form* (3)–(5) *if each external eigenvalue $\lambda_j \notin \boldsymbol{\lambda}_I$ satisfies the following condition*

$$\lambda_j \neq \langle \mathbf{p}, \boldsymbol{\lambda}_I \rangle, \tag{6}$$

for all integer vectors $\mathbf{p} \geq 0$, $\mathbf{p} \in \mathbb{Z}^k$.

Let consider an analytic Hamiltonian system

$$\dot{\mathbf{x}} = \frac{\partial H}{\partial \mathbf{y}}, \quad \dot{\mathbf{y}} = -\frac{\partial H}{\partial \mathbf{x}} \tag{7}$$

with n degrees of freedom near its stationary point $\mathbf{x} = \mathbf{y} = 0$. The eigenvalues of the matrix A can be reordered in a such way: $\lambda_{j+n} = -\lambda_j$, $j = 1, \dots, n$. Denote by $\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_n)$. There exists [3, § 12] a canonical formal transformation which reduces the initial system (7) into its *normal form*

$$\dot{\mathbf{u}} = \partial h / \partial \mathbf{v}, \quad \dot{\mathbf{v}} = -\partial h / \partial \mathbf{u} \tag{8}$$

defined by the normalized Hamiltonian $h(\mathbf{u}, \mathbf{v})$

$$h(\mathbf{u}, \mathbf{v}) = \sum_{j=1}^{n} \lambda_j u_j v_j + \sum h_{\mathbf{pq}} \mathbf{u}^{\mathbf{p}} \mathbf{v}^{\mathbf{q}} \tag{9}$$

containing only resonant terms $h_{\mathbf{pq}}\mathbf{u}^{\mathbf{p}}\mathbf{v}^{\mathbf{q}}$ with

$$\langle \mathbf{p} - \mathbf{q}, \boldsymbol{\lambda} \rangle = 0. \tag{10}$$

By L_I we denote the *coordinate subspace* $L_I = \{\mathbf{u}, \mathbf{v} : u_j = v_j = 0 \text{ for all } j \notin I\}$.

Problem. Which subspaces L_I are invariant in the normal form (8), (9), (10)?

Theorem 2. *The coordinate subspace L_I of dimension $2k$ is invariant in the Hamiltonian normal form if each external eigenvalue $\lambda_j \notin \boldsymbol{\lambda}_I$ satisfies the following condition*

$$\lambda_j \neq \langle \mathbf{p}, \boldsymbol{\lambda}_I \rangle , \tag{11}$$

for any integer vector $\mathbf{p} \neq 0$, $\mathbf{p} \in \mathbb{Z}^k$.

The principal difference between condition (6) in Theorem 1 and condition (11) in Theorem 2 is that any non-zero vector $\mathbf{p}$ is taken from the lattice $\mathbb{Z}^k$ in the Hamiltonian case but it is taken for $0 \leq \mathbf{p} \in \mathbb{Z}^k$ in the general case.

2. Resonance finding by q-analogue of subdiscriminants

It is immediately follows from (6) and (11) that resonant relations can be determined by eigenvalues of linear part (2). Let consider an important case when all the eigenvalues $\boldsymbol{\lambda}$ are either real or pure imaginary.

Here we propose the following algorithm [4] of searching for resonant relations which essential use technique of q-subdiscriminants [5] and elimination theory. Lets denote by q the commensurability of a pair of eigenvalues: $q = \lambda_i/\lambda_j$.

Step 1: Matrix A of linear system (2) is found and its characteristic polynomial $f_n(\lambda)$ is computed.

Step 2: Compute the sequence of k-th q-subdiscriminants $D_q^{(k)}(f_n)$, $0 \leq k \leq n-2$, which are polynomials in coefficients of f_n and q.

Step 3a: If q-discriminant $D_q(f_n)$ as a polynomial in annulus $\mathbb{Z}[q]$ can be factorized then it is possible to find out all the pairs of resonant eigenvalues.

Step 3b: If the previous step fail then a kind of a brute force algorithm can be applied: for mutually prime pairs (r, s) of integers we check the equality $D_q(f_n) = 0$ for $q = r/s$.

Step 4: Let for a certain value $q^* \in \mathbb{Q}$ it is true that $D_{q^*}(f_n) = 0$. Then it is possible with the help of q-subdiscriminants from the Step 2 to determine the structure of eigenvalues with commensurability q^* and in some cases even to compute them.

Step 5: Having all the commensurable eigenvalues λ_i it is possible to check either conditions (6), (11) take place or no.

3. Example

Let all $\lambda_j/\lambda_1 \notin \mathbb{Z}$, $j = 2, \ldots, n$. Then the normal form has two-dimensional invariant subspace $L_1 = \{u_j = v_j = 0, j \notin I_1\}$, where $I_1 = \{1\}$. On the subspace L_1 Hamiltonian NF (9) induces a NF with one degree of freedom and the normalizing transformation converges.

If $\lambda_1 \neq 0$ and is purely imagine, then for real Hamiltonian system the real subspace L_1 is a family of periodic solutions. This fact was found by A.M. Lyapunov in 1892 and was described with the help of Hamiltonian formalism by C. Siegel [6, §§16, 17].

Acknowledgment

The work is supported by RFBR, Project No. 18-01-00422a.

References

[1] A. D. Bruno. *The Restricted 3–body Problem: Plane Periodic Orbits.* Walter de Gruyter, Berlin, 1994. = Nauka, Moscow, 1990. 296 p. (in Russian).

[2] A. D. Bruno. Analytical form of differential equations (I). *Trans. Moscow Math. Soc.*, 25:131–288, 1971. = Trudy Moskov. Mat. Obsc. 25 (1971) 119-262 (in Russian).

[3] A. D. Bruno. Analytical form of differential equations (II). *Trans. Moscow Math. Soc.*, 26:199–239, 1972. = Trudy Moskov. Mat. Obsc. 25 (1971) 119-262 (in Russian).

[4] A. B. Batkhin. Invariant coordinate subspaces of normal form of a system of ordinary differential equations. *Preprints of KIAM*, (72), 2020. (in Russian). `https://doi.org/10.20948/prepr-2020-72` `doi:10.20948/prepr-2020-72`.

[5] A. B. Batkhin. Parameterization of a set determined by the generalized discriminant of a polynomial. *Programming and Computer Software*, 44(2):75–85, 2018. `https://doi.org/10.1134/S0361768818020032` `doi:10.1134/S0361768818020032`.

[6] C. L. Siegel and J. K. Moser. *Lectures on Celestial Mechanics.* Springer-Verlag, Berlin, Heidelber, New York, 1971.

Alexander Batkhin
Department of Singular Problems
Keldysh Institute of Applied Mathematics of RAS
Department of Theoretical Mechanics
Moscow Institute of Physics and Technology
Moscow, Russia
e-mail: `batkhin@gmail.com`