Invariant coordinate subspaces of normal form of Hamiltonian system

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Introduction (1)

An approach of Poincaré for investigation of systems of nonlinear ordinary differential equations was based on the maximal simplification of the right-hand sides of these equations by invertible transformations. This approach led to the theory of normal forms (NF) of the general system and in particularly of the Hamiltonian ones.

The theory of NF for general systems was was developed by A.Dulac & A.D.Bruno [Bruno, 1971] and in the Hamiltonian case by G.D. Birkhoff, T.M. Cherry, F.G. Gustavson, C.L. Siegel, J. Moser, A.D. Bruno and others (see [Bruno, 1994, Chs. I, II] for more details).

Introduction (2)

Even though the NF is a formal object it can be used for

1. studing stability and bifurcations [Bruno, 1989],;
2. checking local integrability [Bruno (et al.), 2017];
3. searching first integrals, families of periodic solutions of the system [Bruno, 2020a,b];
4. asymptoting integration of the system.

The existance of invariant subspace can reduce the phase flow on the space of less dimension and in some cases can give information about periodic solution of the whole system.

Introduction (3)

The goal of the presented work is to

1. investigate invariant coordinates subspaces of NF of a general and Hamiltonian system with non-degenerated linear part;
2. present a method of searching of resonance relations between eigenvalues of linear part of the system without their explicit computation.

The talk is based on my last preprint
Batkhin A. B. Invariant coordinate subspaces of normal form of a system of ordinary differential equations. // Preprints of KIAM. 2020. No. 72. (in Russian)

Remark on notations

- Boldface symbols like $\mathbf{x}, \mathbf{y}, \mathbf{u}, \mathbf{v}$ denote column-vectors in n-dimensional real $\mathbb{R}^n$ or complex $\mathbb{C}^n$ spaces.
- Boldface symbols like $\mathbf{p}, \mathbf{q}$ denote vectors in n-dimensional integer lattice $\mathbb{Z}^n$.
- $|\mathbf{p}| = \sum_{j=1}^n |p_j|$.
- For $\mathbf{x} = (x_1, \ldots, x_n)^{\mathrm{T}}$ and $\mathbf{p} = (p_1, \ldots, p_n)^{\mathrm{T}}$ we denote by $\mathbf{x}^{\mathbf{p}} \equiv \prod\limits_{j=1}^{n} x_j^{p_j}$ and by $\langle \mathbf{p}, \mathbf{x} \rangle \equiv \sum\limits_{j=1}^{n} p_j x_j$.

1. Introduction

2. Invariant subspaces of general and Hamiltonian ODE systems
- Invariant subspaces of general ODE system
- Invariant subspaces of a Hamiltonian system
- Examples

3. Searching for resonant relations by q-analogue of subdiscriminants
- Method of searching for resonant relations
- Model example

Linear part of system of general ODE

Consider an analytical system of ODE

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) \tag{1}$$

near its stationary point $\mathbf{x} = 0$.

Let the linear part

$$\dot{\mathbf{x}} = A\mathbf{x}, \quad A = \left.\frac{\partial \mathbf{f}}{\partial \mathbf{x}}\right|_{\mathbf{x}=0},$$

of the system (1) be non-degenerated. Let matrix A has n eigenvalues at least one of which is non-zero $\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_n)$.

Normal form (1)

There exists [Bruno, 1971] a formal invertible transformation $\mathbf{g} : \mathbf{x} \leftrightarrow \mathbf{y}$

$$\mathbf{x} = \mathbf{g}(\mathbf{y}),$$

represented in the form of power series, which reduces the initial system (1) into its ***normal form***

Normal form (2)

Definition

Normal form of the initial system (1) is a system of the form

$$\dot{y}_j = y_j h_j(\mathbf{y}), \quad j = 1, \dots, n, \tag{2}$$

right-hand sides $y_j h_j(\mathbf{y})$ of which are power series

$$y_j h_j(\mathbf{y}) = y_j \sum_{\mathbf{q}} h_{j\mathbf{q}} \mathbf{y}^{\mathbf{q}}, \quad h_{j0} = \lambda_j, \quad j = 1, \dots, n, \tag{3}$$

containing only resonant terms with

$$\langle \mathbf{q}, \boldsymbol{\lambda} \rangle = 0. \tag{4}$$

Here $h_{j\mathbf{q}}$ are constant coefficients and in $y_j h_j(\mathbf{y})$ coordinate $q_j \geqslant -1$, but others $q_k \geqslant 0$.

Coordinate subspace

Let $I = \{i_1, \dots, i_k\}$ be a set of increasing indices $1 \leqslant i_1$, $i_k \leqslant n$, $k \leqslant n$. By K_I we denote the ***coordinate subspace***

$$K_I = \{\mathbf{y} : y_j = 0 \text{ for all } j \notin I\}.$$

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The eigenvalues λ_j, $j \in I$, corresponding to the internal coordinates $\mathbf{y}_I$ we call ***internal eigenvalues*** and denote them by $\boldsymbol{\lambda}_I$. Others λ_j, $j \notin I$, are called ***external eigenvalues***.

Problem 1

Which subspaces K_I are invariant in the normal form (2), (3), (4)?

Theorem 1.

The coordinate subspace K_I of dimension k is invariant in the normal form (2)–(4) if each external eigenvalue $\lambda_j \notin \boldsymbol{\lambda}_I$ satisfies the following condition

$$\lambda_j \neq \langle \mathbf{p}_I, \boldsymbol{\lambda}_I \rangle \,, \tag{5}$$

for all integer vectors $\mathbf{p}_I \geqslant 0$, $\mathbf{p}_I \in \mathbb{Z}^k$.

Proof of Theorem 1

Under condition (5) it follows that each series $h_j(\mathbf{y})$ for $j \notin I$ does not contain any term $h_{j\mathbf{q}}\mathbf{y}^{\mathbf{q}}$ with indices $q_j = -1$, $q_i \geqslant 0$, $i \neq j \notin I$.

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Since the external variables y_j, $j \notin I$, are equal to zero in the subspace K_I, it follows that for $\mathbf{y} \in K_I$

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So the subspace K_I is invariant in the normal form (2), (3), (4). $\square$

Hamiltonian system near equilibrium

We consider an analytic Hamiltonian system

$$\dot{\mathbf{x}} = \frac{\partial H}{\partial \mathbf{y}}, \quad \dot{\mathbf{y}} = -\frac{\partial H}{\partial \mathbf{x}} \tag{6}$$

with n degrees of freedom near its stationary point

$$\mathbf{x} = \mathbf{y} = 0.$$

The Hamiltonian function $H(\mathbf{x}, \mathbf{y})$ is expanded into convergent power series

$$H(\mathbf{x}, \mathbf{y}) = \sum H_{\mathbf{pq}} \mathbf{x}^{\mathbf{p}} \mathbf{y}^{\mathbf{q}}$$

with constant coefficients $H_{\mathbf{pq}}$, $\mathbf{p}, \mathbf{q} \geqslant 0$, $|\mathbf{p}| + |\mathbf{q}| \geqslant 2$.

Linear part of the Hamiltonian system (1)

Canonical transformations of coordinates $\mathbf{x}, \mathbf{y}$

$$\mathbf{x} = \mathbf{f}(\mathbf{u}, \mathbf{v}), \quad \mathbf{y} = \mathbf{g}(\mathbf{u}, \mathbf{v}), \tag{7}$$

preserve the Hamiltonian character of the initial system (6).

Let denote by $\mathbf{z} = (\mathbf{x}, \mathbf{y}) \in \mathbb{R}^{2n}(\mathbb{C}^{2n})$ be the phase vector. Then the linear part of the system (6) can be written in the form

$$\dot{\mathbf{z}} = B\mathbf{z}, \quad B = \frac{1}{2}\begin{pmatrix} \frac{\partial^2 H}{\partial \mathbf{y}\partial \mathbf{x}} & \frac{\partial^2 H}{\partial \mathbf{y}\partial \mathbf{y}} \\ -\frac{\partial^2 H}{\partial \mathbf{x}\partial \mathbf{x}} & -\frac{\partial^2 H}{\partial \mathbf{x}\partial \mathbf{y}} \end{pmatrix}\Bigg|_{\mathbf{x}=\mathbf{y}=0}. \tag{8}$$

Let $\lambda_1, \ldots, \lambda_{2n}$ be eigenvalues of the matrix B, which can be reordered in a such way that $\lambda_{j+n} = -\lambda_j$, $j = 1, \ldots, n$. Denote by $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_n)$.

Hamiltonian normal form

There exists [Bruno, 1972, § 12] a canonical formal transformation (7), where all $\mathbf{f}$ and $\mathbf{g}$ are power series, which reduces the initial system (6) into its ***normal form***

$$\dot{\mathbf{u}} = \partial h/\partial \mathbf{v}, \quad \dot{\mathbf{v}} = -\partial h/\partial \mathbf{u} \tag{9}$$

defined by the normalized Hamiltonian $h(\mathbf{u}, \mathbf{v})$

$$h(\mathbf{u}, \mathbf{v}) = \sum_{j=1}^{n} \lambda_j u_j v_j + \sum h_{\mathbf{pq}} \mathbf{u}^{\mathbf{p}} \mathbf{v}^{\mathbf{q}} \tag{10}$$

containing only resonant terms $h_{\mathbf{pq}} \mathbf{u}^{\mathbf{p}} \mathbf{v}^{\mathbf{q}}$ with

$$\langle \mathbf{p} - \mathbf{q}, \boldsymbol{\lambda} \rangle = 0. \tag{11}$$

Here $0 \leqslant \mathbf{p}, \mathbf{q} \in \mathbb{Z}^n$, $|\mathbf{p}| + |\mathbf{q}| \geqslant 2$ and $h_{\mathbf{pq}}$ are constant coefficients.

Coordinate subspace

Let $I = \{i_1, \ldots, i_k\}$ be a set of increasing indices $1 \leqslant i_1,\ i_k \leqslant n,\ k \leqslant n$. By L_I we denote the ***coordinate subspace***

$$L_I = \{\mathbf{u}, \mathbf{v} : u_j = v_j = 0 \text{ for all } j \notin I\}\,.$$

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Problem 2

Which subspaces L_I are invariant in the normal form (9), (10), (11)?

Theorem 2.

The coordinate subspace L_I of dimension $2k$ is invariant in the normal form (9)–(11) if each external eigenvalue $\lambda_j \notin \boldsymbol{\lambda}_I$ satisfies the following condition

$$\lambda_j \neq \langle \mathbf{p}_I, \boldsymbol{\lambda}_I \rangle , \tag{12}$$

for any integer vector $\mathbf{p}_I \neq 0$, $\mathbf{p}_I \in \mathbb{Z}^k$.

Proof.

It is evident consequence of Theorem 1. □

Remark

The principal difference between condition (5) in Theorem 1 and condition (12) in Theorem 2 is that any non-zero vector $\mathbf{p}$ is taken from the lattice $\mathbb{Z}^k$ in the Hamiltonian case but it is taken for $0 \leqslant \mathbf{p} \in \mathbb{Z}^k$ in the general case.

Example 3.1

Let eigenvalues of linear part of general ODE system equal to

$$\lambda_1 = 1, \lambda_2 = \sqrt{2}, \lambda_3 = 1 + \sqrt{2}.$$

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There are 3 1-D invariant subspaces K_j, $j = 1{,}2{,}3$, correponding to each eigenvalue λ_j, $j = 1{,}2{,}3$, because for any $i \neq j$ one has $\lambda_i/\lambda_j \notin \mathbb{N}$.

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2-D subspace K_{12} is not invariant due to resonant relation $\lambda_3 = \lambda_1 + \lambda_2$, but others K_{13} and K_{23} are invariant.

Example 3.2

Let all $\lambda_j/\lambda_1 \notin \mathbb{Z}$, $j = 2, \ldots, n$. Then the normal form has two-dimensional invariant subspace $L_1 = \{u_j = v_j = 0, j \notin I_1\}$, where $I_1 = \{1\}$. On the subspace L_1 normal form (10) induces a Hamiltonian normal form with one degree of freedom and the normalizing transformation converges.

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If $\lambda_1 \neq 0$ and is purely imagine, then for real Hamiltonian system (6) the real subspace L_1 is a family of periodic solutions. This fact was found by A.M. Lyapunov [1892] and was described with the help of Hamiltonian formalism by C. Siegel [Siegel (et al.), 1971, §§16, 17].

Example 3.3 (1)

Let there exists the only pair of eigenvalues λ_1, λ_2 with property

$$\lambda_2/\lambda_1 = r/s$$

where $r, s \in \mathbb{N}$, i.e. the multiplicity of the resonance is equal to 1.

The resonant equation (11) $\langle \mathbf{p} - \mathbf{q}, \boldsymbol{\lambda} \rangle = 0$ has two kinds of solutions, which correspond to two types of resonant terms:

- secular terms, for $\mathbf{p} = \mathbf{q}$, which always exists due to special structure of matrix B of linearized system (8);
- pure resonant terms, which corresponds to nontrivial integer solutions of the equation $\langle \mathbf{p}, \boldsymbol{\lambda} \rangle = 0$.

Example 3.3 (2)

Let for instance the number $s = 1$. Then the normal form (10) contains the resonant terms $u_1^r v_2$ and $v_1^r u_2$. It means that the subspace L_1 can be not invariant, because the right-hand sides of equations for variables u_2, v_2 have terms depending on variables u_1, v_1 and these right-hand sides cannot be always equal zero for the case $u_2 = v_2 = 0$.

Example 3.4

Let there are four pairs of eigenvalues of a Hamiltonian system: one pair of real ± 1, second one of pure imagine $\pm\mathrm{i}$ and the third and fourth ones of complex $\pm 1 \pm \mathrm{i}$. Lets reorder these eigenvalues in a such manner:

$$\lambda_1 = 1, \quad \lambda_2 = \mathrm{i}, \quad \lambda_3 = 1 + \mathrm{i}, \quad \lambda_4 = 1 - \mathrm{i}.$$

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There are ***six*** subspaces of dimension 4:

L_{12} is not invariant, because the external eigenvalue $\lambda_3 = \lambda_1 + \lambda_2$.
L_{13} is not invariant, because the external eigenvalue $\lambda_2 = \lambda_3 - \lambda_1$.
L_{14} is not invariant, because the external eigenvalue $\lambda_2 = \lambda_1 - \lambda_4$.
L_{23} is not invariant, because the external eigenvalue $\lambda_1 = \lambda_3 - \lambda_2$.
L_{24} is not invariant, because the external eigenvalue $\lambda_1 = \lambda_2 + \lambda_4$.

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L_{24} is not invariant, because the external eigenvalue $\lambda_1 = \lambda_2 + \lambda_4$.
L_{34} is ***invariant***, because neither λ_1 nor λ_2 cannot be obtained as a linear combination of λ_3 and λ_4 with integer coefficients. Thus the condition (12) of Theorem 2 is satisfied.

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Finally, there are ***no*** any invariant subspace of dimension 6.

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   - Method of searching for resonant relations
   - Model example

Searching for resonant relations by q-analogue of subdiscriminants

It is immediately follows from conditions of Theorems (1) and (2) that resonant relations can be determined by eigenvalues of linear part. Let consider an important case when all the eigenvalues $\boldsymbol{\lambda}$ are either real or pure imaginary.

Here we propose the method [Batkhin, 2020] of searching for resonant relations which essential use technique of q-subdiscriminants [Batkhin, 2018; 2019] and elimination theory. Lets denote by q the commensurability of a pair of eigenvalues: $q = \lambda_i/\lambda_j$.

Method of searching for resonant relations

Step 1 Matrix A of linear system is found and its characteristic polynomial $f_n(\lambda)$ is computed.

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Step 3b If the previous step fail then a kind of a brute force method can be applied: for mutually prime pairs (r, s) of integers limited by a certain value $|r|+|s| \le m$ we check the equality $D_q(f_n) = 0$ for $q = r/s$.

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Step 4 Let for a certain value $q^* \in \mathbb{Q}$ it is true that $D_{q^*}(f_n) = 0$. Then it is possible with the help of q-subdiscriminants from the Step 2 to determine the structure of eigenvalues with commensurability q^* and in some cases even to compute them.

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Step 5 Having all the commensurable eigenvalues λ_i it is possible to check either conditions (5), (12) take place or no.

Model example

Consider 6 identical pendulums of length l and mass m, which are connected to each other with weightless linearly elastic springs of rigidity k length d in the unperturbed state. Spring attachment points are located at a distance of $b \leqslant d$ from the pendulum suspension points.
These systems moves on the vertical plane and has 6 DOF.

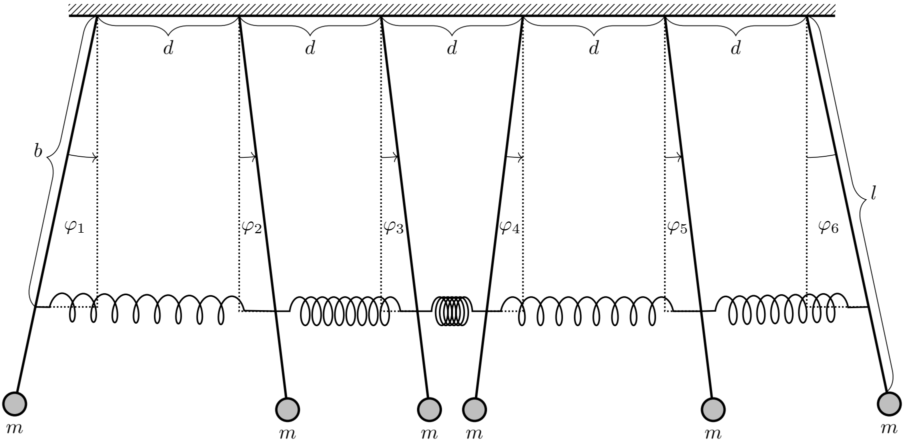

Figure 1: Six identical pendulums with springs

Model example (1)

Characteristic polynomial of linear part can be written in the form

$$\begin{aligned}\hat{f}_6(\mu) = &\mu^6 - 2\,(5\beta+3)\,\mu^5 + \left(36\beta^2+50\beta+15\right)\mu^4 - \\ &- 2\left(28\beta^3+72\beta^2+50\beta+10\right)\mu^3 + \\ &+ \left(35\beta^4+168\beta^3+216\beta^2+100\beta+15\right)\mu^2 - \\ &- 2\left(3\beta^5+35\beta^4+84\beta^3+72\beta^2+25\beta+3\right)\mu + \\ &+ (2\beta+1)(3\beta+1)(\beta+1)(\beta^2+4\beta+1).\end{aligned}$$

where $\mu = \lambda^2$, and the only parameter is $\beta = \dfrac{kb^2}{mgl} > 0$.

In CAS `Maple` and `Mathematica` q-discriminant $D_q\left(\hat{f}_6\right)$ can be simplified and factorized into linear and quadratic factors. So it is possible to find all the values of q for which rationally commensurable roots exists.

Model example (2)

In CAS SymPy q-discriminant $D_q\left(\hat{f}_6\right)$ cannot be factorized, so brute force method was applied for the certain value $\beta = 48/25$.

The maximal resonance order is $m = 20$ and a tuple of pairs (r^2, s^2) such that $\mathrm{GCD}\,(r, s) = 1$ and $r + s \leqslant 20$, $r, s \in \mathbb{N}$ was constructed. Two zeroes $q_1^* = 121/25$ and $q_2^* = 169/25$ of $q$discriminant were found. Consequently, there must be another value $q_3^* = q_2^*/q_1^* = 169/121$.

Thus we have resonant relation

$$5p_1 + 11p_2 + 13p_3 = 0 \tag{13}$$

between eigenvalues $\lambda_1 = 1, \lambda_2 = 11/5, \lambda_3 = 13/5$.

Model example (3)

Relation (13) has three integer solutions

$$\mathbf{p}_1 = (1, -4, 3), \quad \mathbf{p}_2 = (3, 1, -2), \quad \mathbf{p}_3 = (4, -3, 1).$$

It means that resonant condition (12) of Theorem 2 does not take place.

The system has 6-D invariant coordinate subspace in its normal coordinates L_I, $I = \{1,2,3\}$, corresponding to the eigenvalues $\lambda_1, \lambda_2, \lambda_3$. Other eigenvalues λ_i, $i = 4,5,6$, give three 2-D subspaces L_i.

Due to the fact that the orders of integer vectors $\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3$ are equal to 8, 6, 8, respectively, the investigation of the source system dynamics in the subspace L_I is possible only if the nonlinear normalization of the source system is performed at least up to 6-th order inclusive.

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References (1)

- *Batkhin A. B.* Parameterization of a Set Determined by the Generalized Discriminant of a Polynomial. // Programming and Computer Software. 2018. Vol. 44, no. 2. P. 75–85. DOI: 10.1134/S0361768818020032.
- *Batkhin A. B.* Computation of the resonance set of a polynomial under constraints on its coefficients. // Programming and Computer Software. 2019. Vol. 45, no. 2. P. 27–36. DOI: https://doi.org/10.1134/S0361768819020038.
- *Batkhin A. B.* Invariant coordinate subspaces of normal form of a system of ordinary differential equations. // Preprints of KIAM. 2020. No. 72. DOI: 10.20948/prepr-2020-72. (in Russian).
- *Bruno A. D.* Analytical form of differential equations (I). // Trans. Moscow Math. Soc. 1971. Vol. 25. P. 131–288. = Trudy Moskov. Mat. Obsc. 25 (1971) 119-262 (in Russian).
- *Bruno A. D.* Analytical form of differential equations (II). // Trans. Moscow Math. Soc. 1972. Vol. 26. P. 199–239. = Trudy Moskov. Mat. Obsc. 25 (1971) 119-262 (in Russian).

References (2)

- *Bruno A. D.* Local Methods in Nonlinear Differential Equations. Berlin – Heidelberg – New York – London – Paris – Tokyo: Springer–Verlag, 1989.
- *Bruno A. D.* The Restricted 3–body Problem: Plane Periodic Orbits. Berlin: Walter de Gruyter, 1994. = Nauka, Moscow, 1990. 296 p. (in Russian).
- *Bruno A. D.* Normal form of a Hamiltonian system with a periodic perturbation. // Computational Mathematics and Mathematical Physics. 2020a. Vol. 60, no. 1. P. 36–52. DOI: 10.1134/S0965542520010066. = Zhurnal Vychislitelnoi Matematiki i Matematicheskoi Fiziki, 2020, V. 60(1): 39–56 (in Russian).
- *Bruno A. D.* Normalization of the periodic Hamiltonian system. // Programming and Computer Software. 2020b. Vol. 46, no. 2. P. 76–83. DOI: 10.1134/S0361768820020048.
- *Bruno A. D., Enderal V. F., Romanovski V. G.* Computer Algebra in Scientific Computing: Proceedings CASC 2017. //. Vol. 10490 / ed. by V. P. Gerdt, et al. Berlin Heidelberg: Springer, 2017. Chap. On new integrals of the Algaba-Gamero-Garcia system. (Lecture Notes in Computer Science). DOI: 10.1007/978-3-642-32973-9.

References (3)

- *Lyapunov A. M.* Problème général de la stabilité du mouvement. // Ann. Fac. Sci. Toulouse Math. 1892. T. 9 (1907), n° 2. P. 204-474. Ann. of Math. Studies (1947). Stability of Motion. New York, London : Acadamic Press (1966) [English].
- *Siegel C. L.*, *Moser J. K.* Lectures on Celestial Mechanics. Berlin, Heidelber, New York: Springer-Verlag, 1971.

Thanks for your attention!