

Numerical Symbolic Dynamics: Complexity of Finite Sequences

(Complexity of trajectories in the equal-mass free-fall three-body problem)

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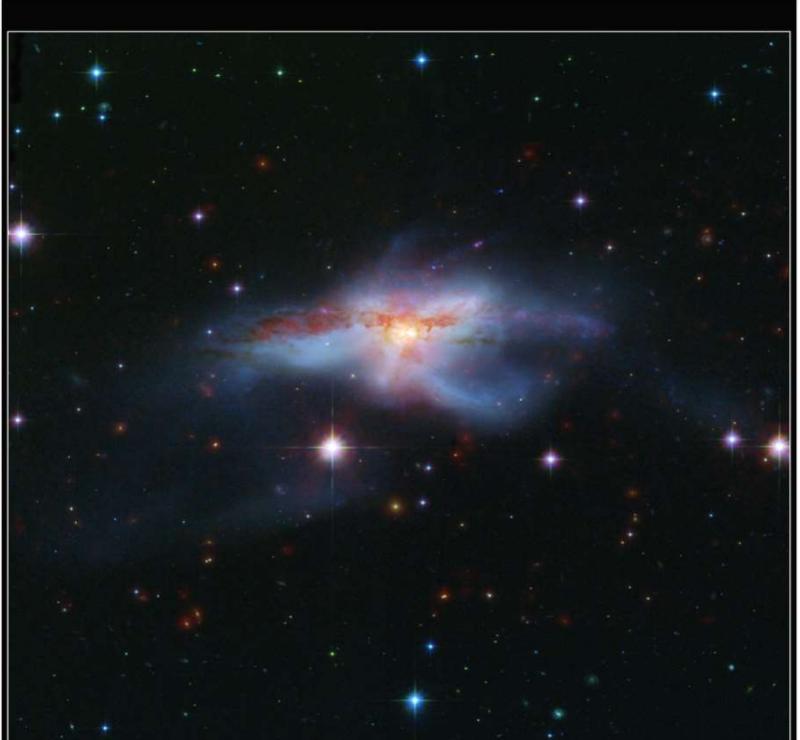
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³⁾ *V.A. Steklov Inst. of Mathematics, St. Petersburg, Russia*



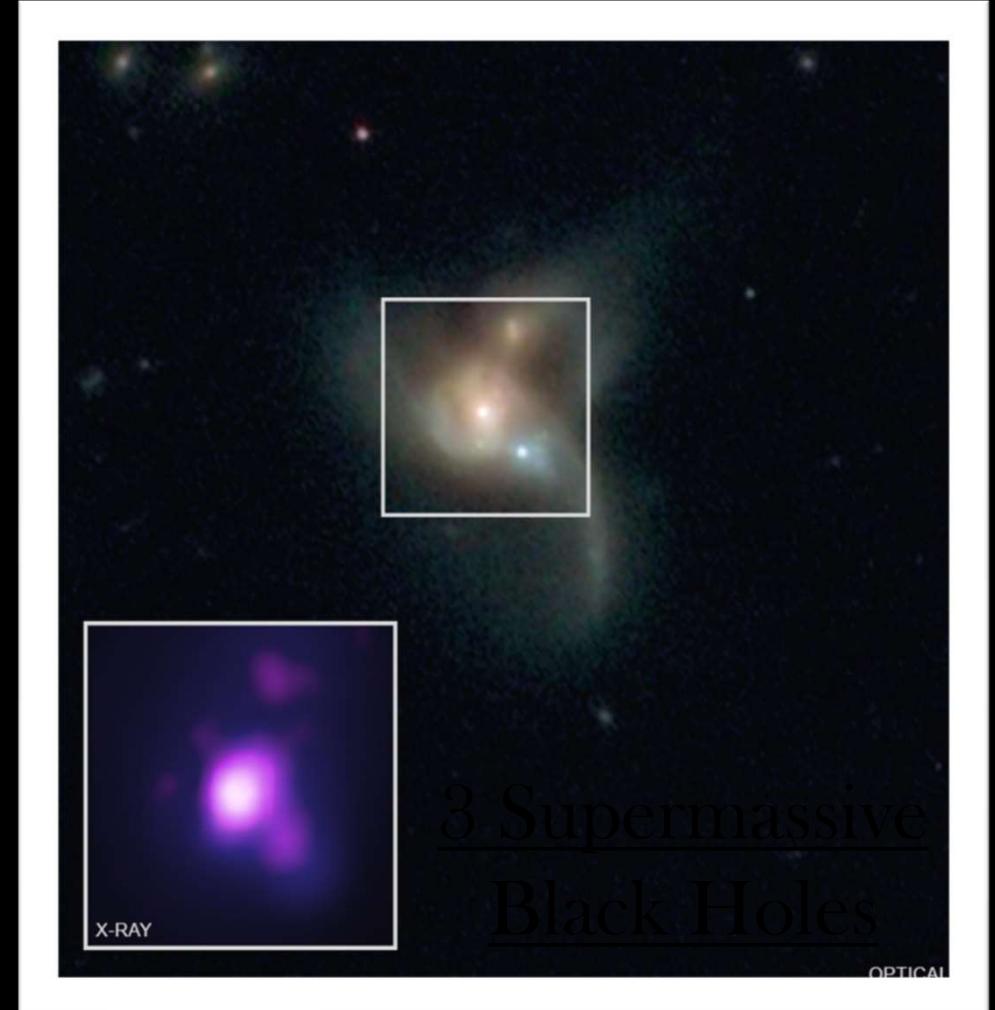
Astronomers have found a distant galaxy with a trio of tightly bound supermassive black holes looming in its central region.



Ultraluminous Galaxy Merger NGC 6240

Spitzer Space Telescope • IRAC
Hubble Space Telescope • ACS

NASA / JPL-Caltech / STScI-ESA / S. Bush (Harvard-Smithsonian CfA) ssc2009-06a



3 Supermassive
Black Holes

OPTICAL

Initial conditions

In the general case:

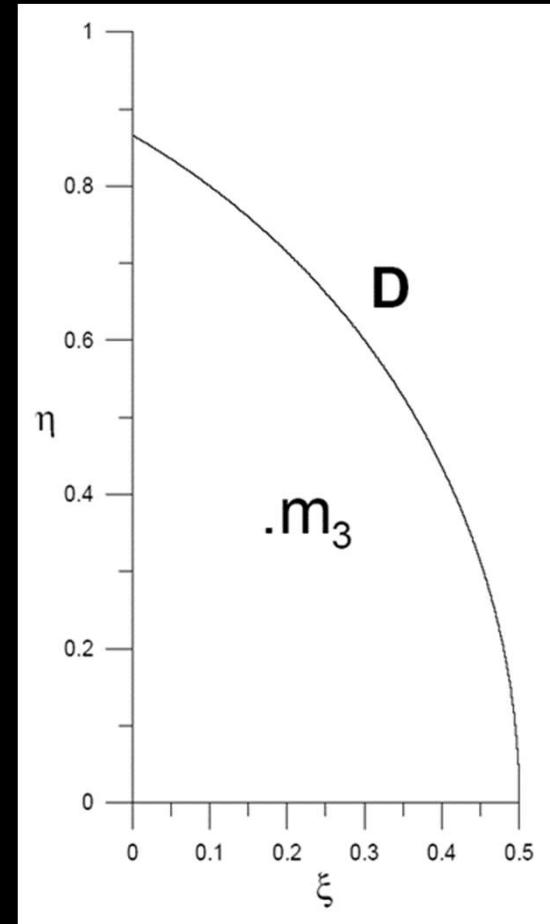
3 masses of the bodies

+

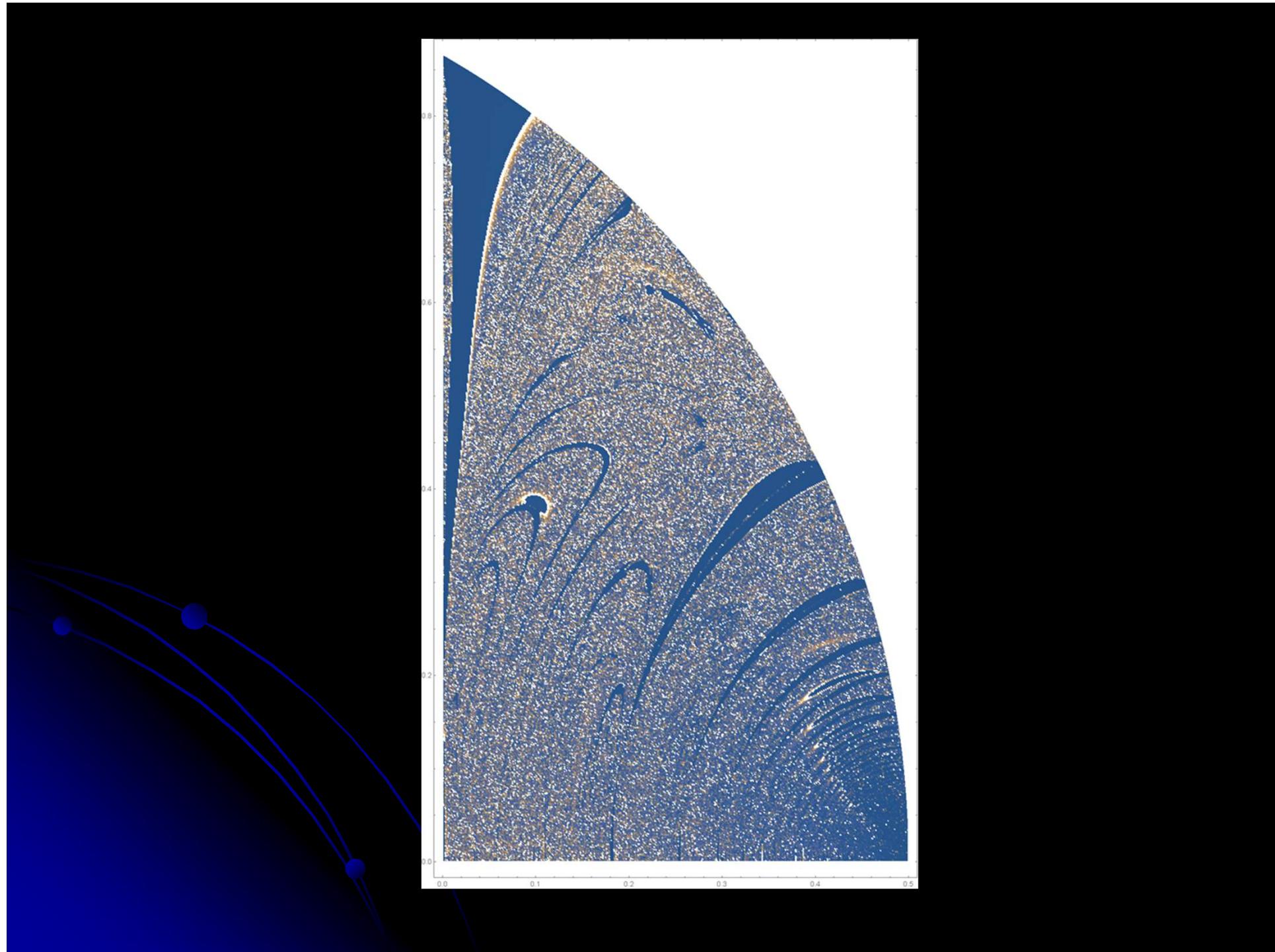
9 initial coordinates

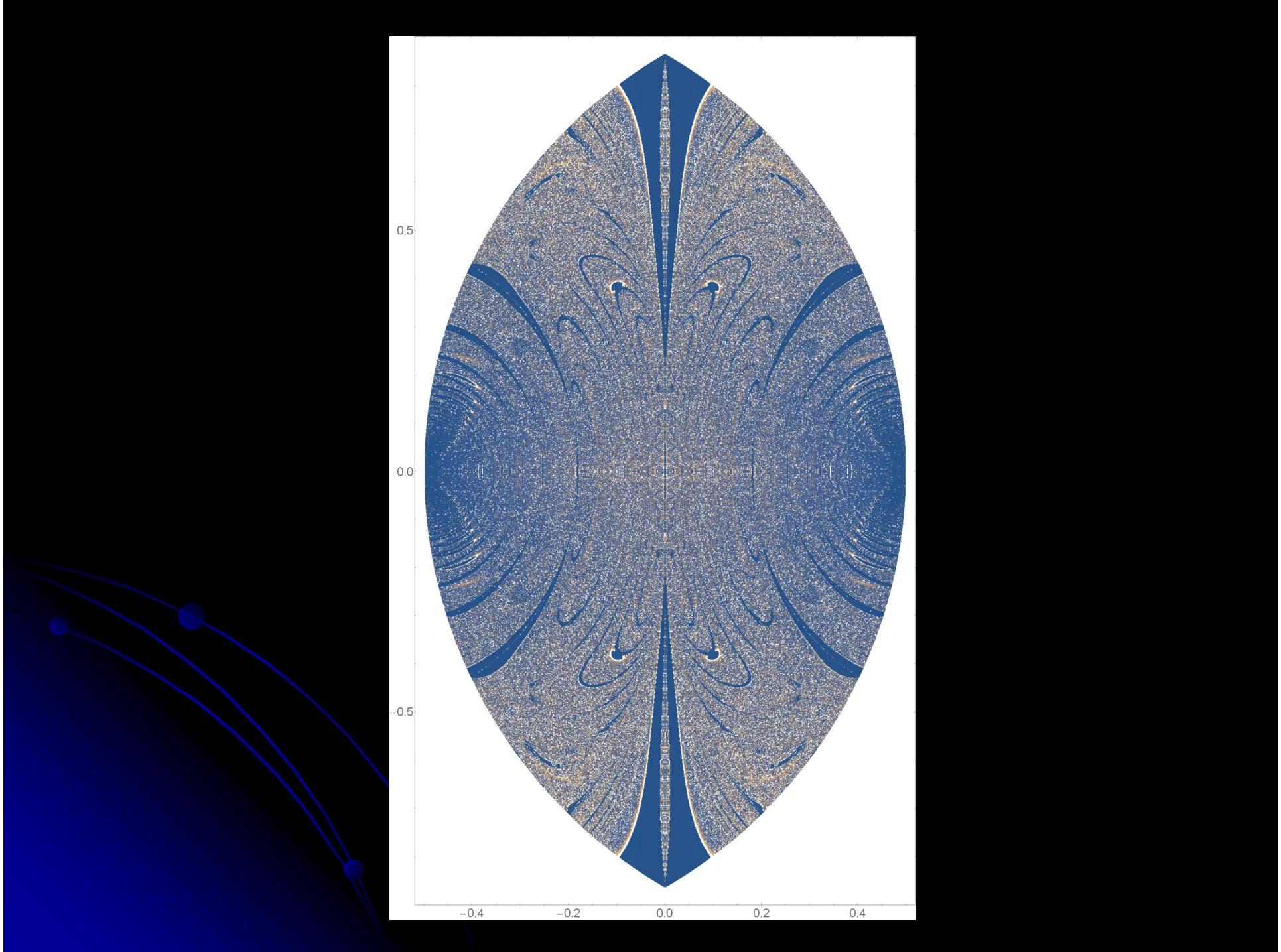
+

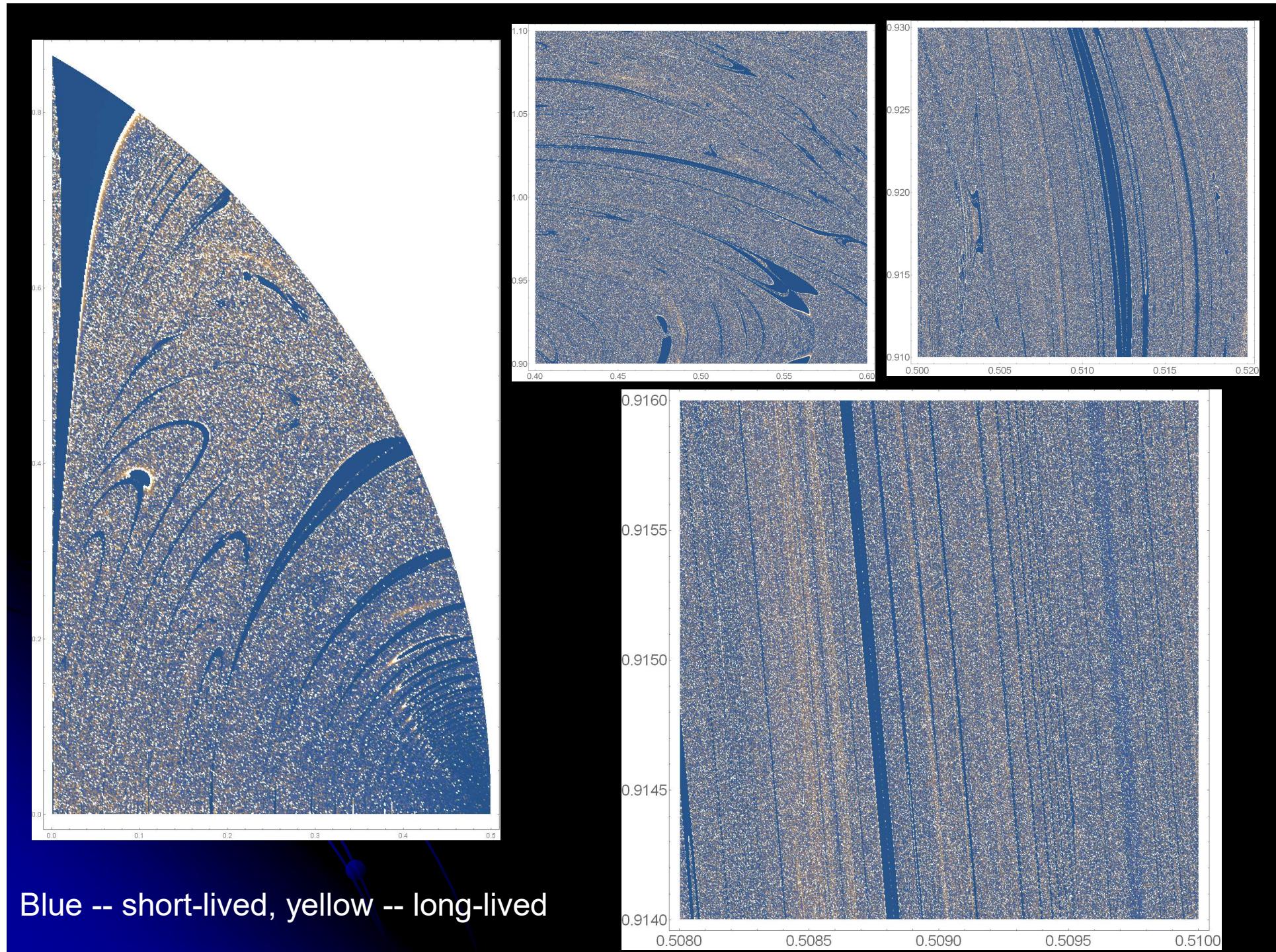
9 initial velocities



Equal-mass free-fall three-body problem:
two (!) coordinates







Different ways to construct the symbolic sequence

- Partitioning of the phase space
- Fixing special dynamical states during the evolution of the triple system
(binary encounters, triple encounters, special configurations, etc.)

Shannon Entropy

$$H_{Sh} = - \sum_i p_i \ln p_i$$

Markov Entropy

$$H^l = - \sum_i p_i \sum_j q_{ij} \log q_{ij}$$

Markov Entropies

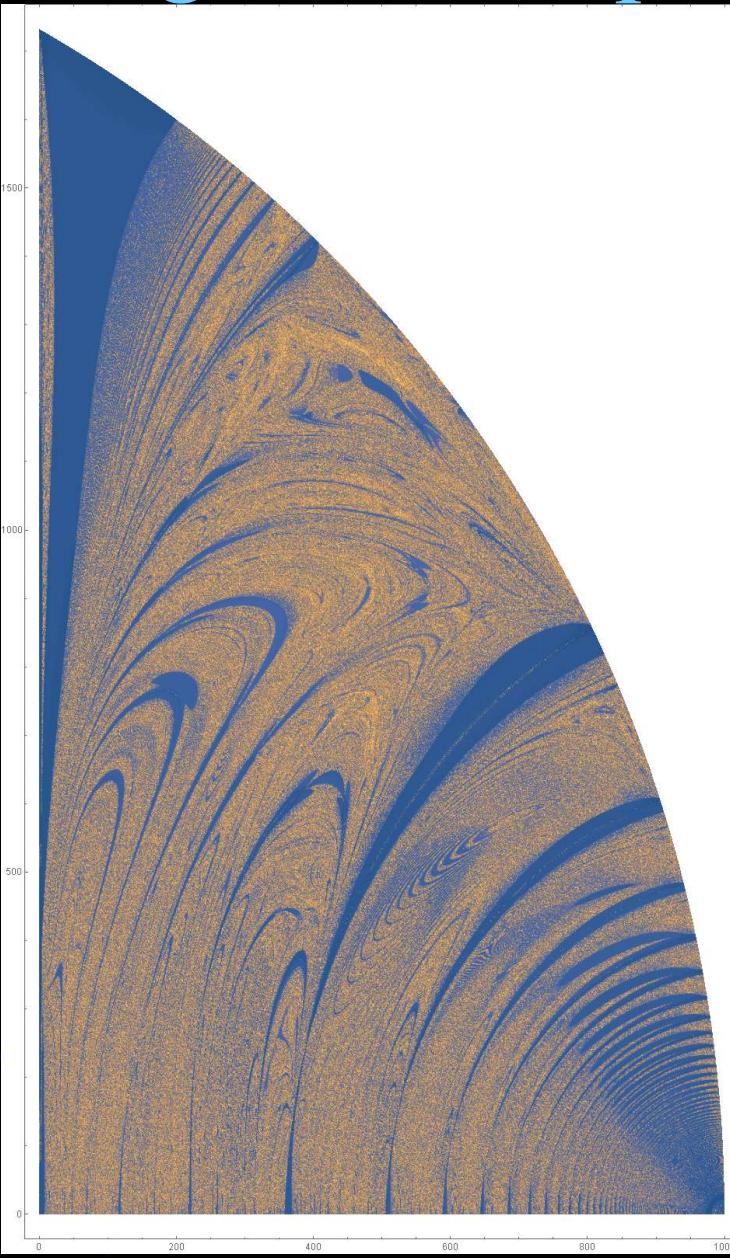
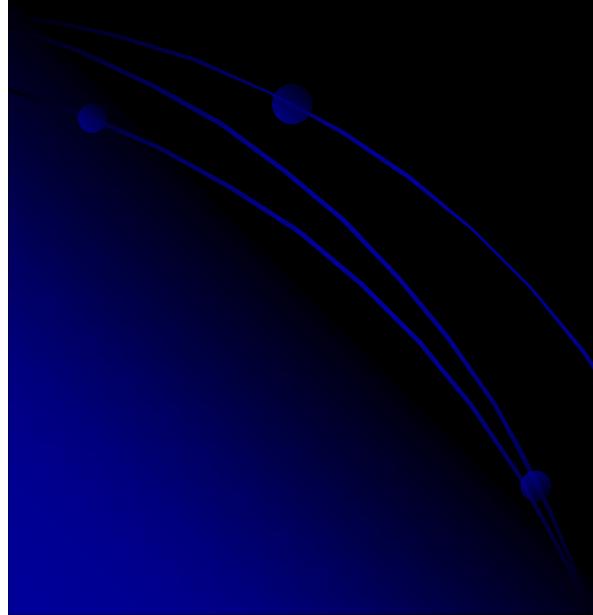
$$H^l = - \sum_i p_i \sum_j q_{ij}^{(l)} \ln q_{ij}^{(l)}$$

p_i – frequency of symbol “ i ” in the sequence;

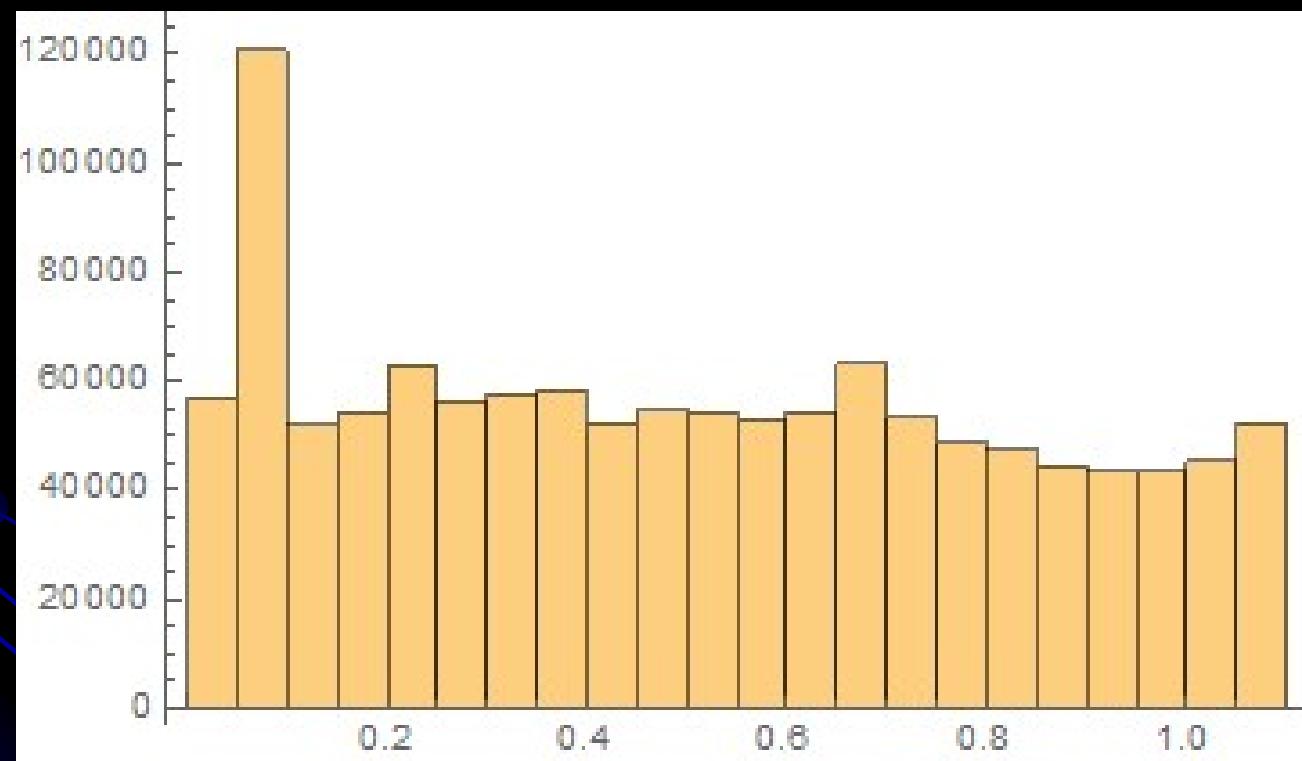
q_{ij} – frequency of transitions from “ i ” to “ j ”;

$q_{ij}^{(l)}$ – \dots $j = i + l.$

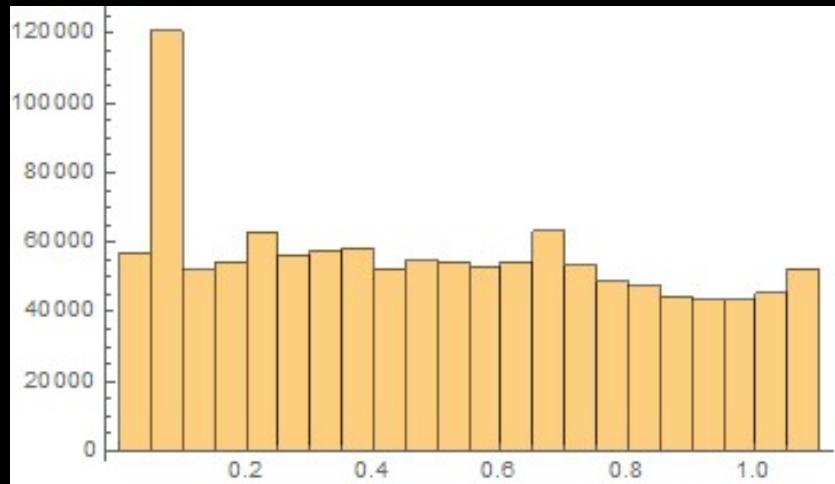
Kolmogorov complexity



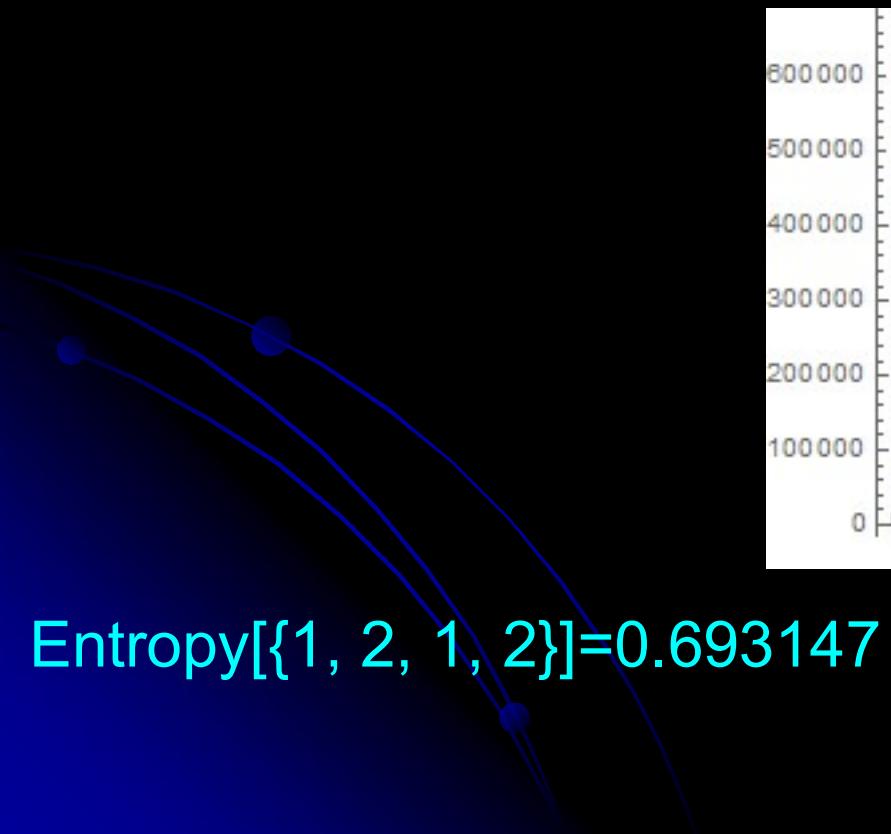
Shannon Entropy



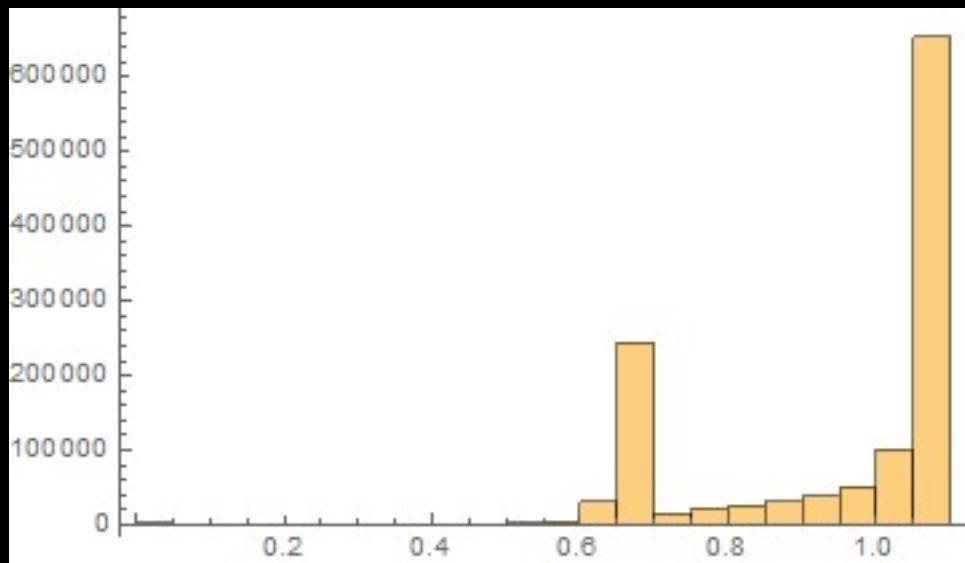
Binary encounters



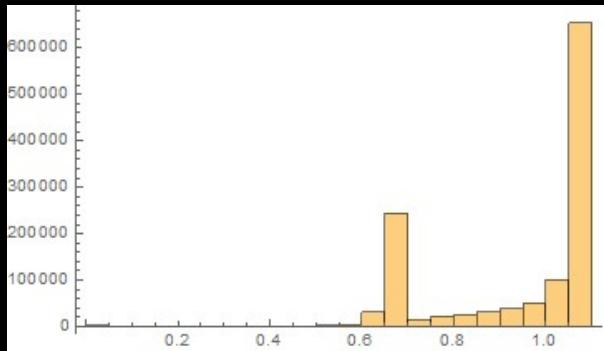
Shannon vs. Shannon Max



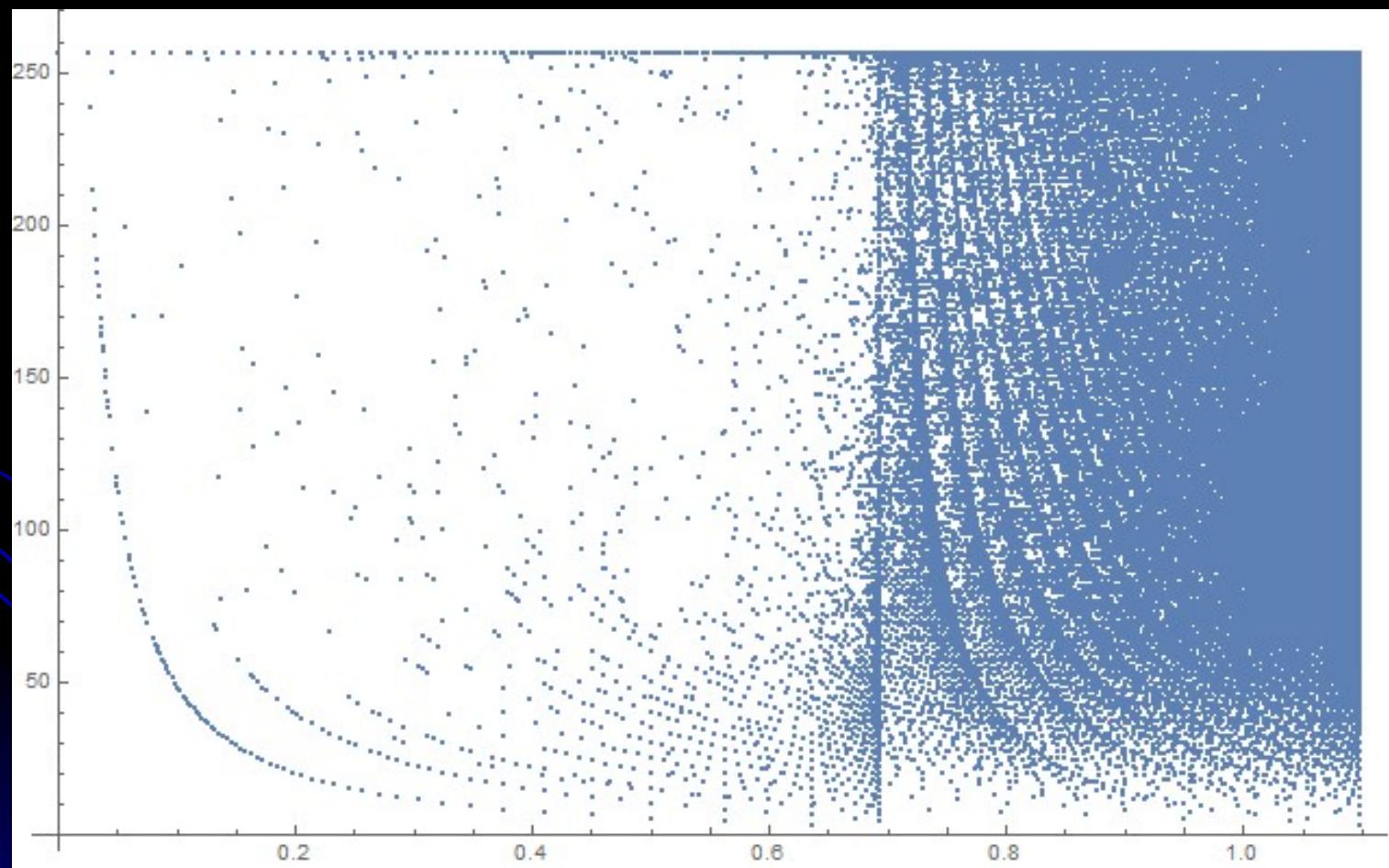
Entropy[$\{1, 2, 1, 2\}$]=0.693147



entropy of $\{1, 2, 3, 1, 2, 3\}$ (or of $\{1, 1, 1, 2, 2, 2, 3, 3, 3\}$) is 1.09861



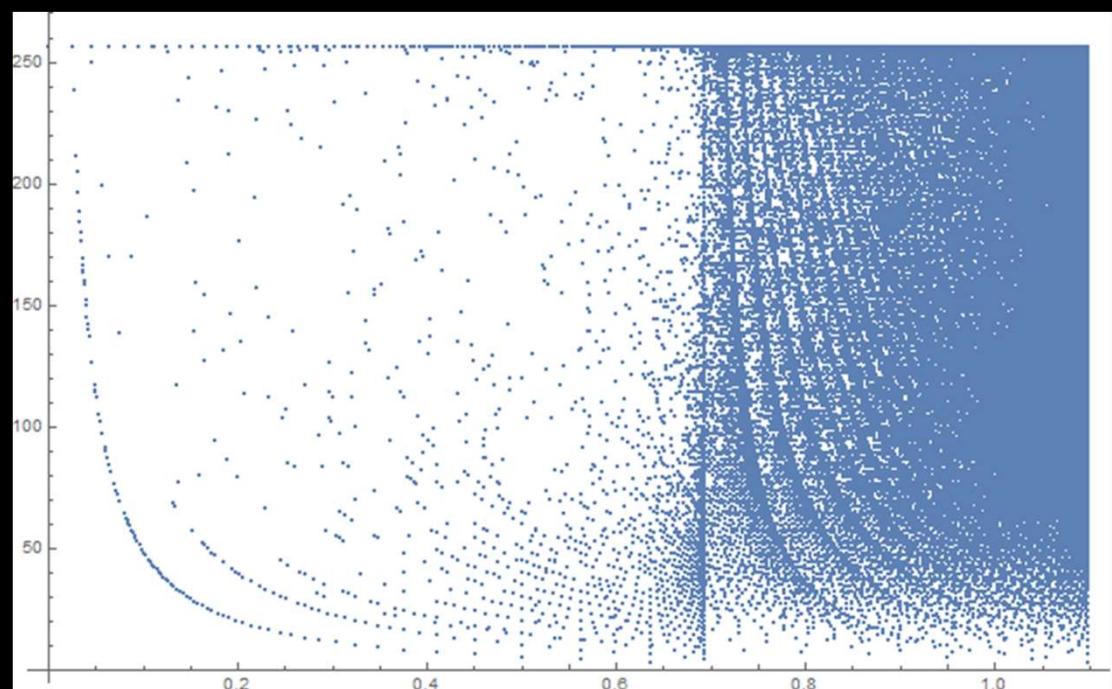
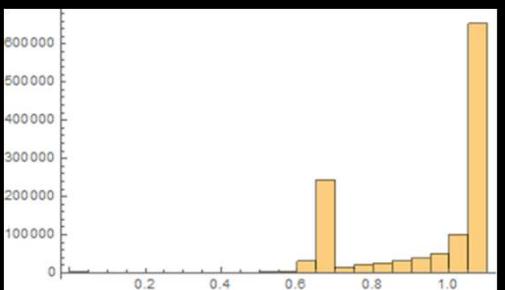
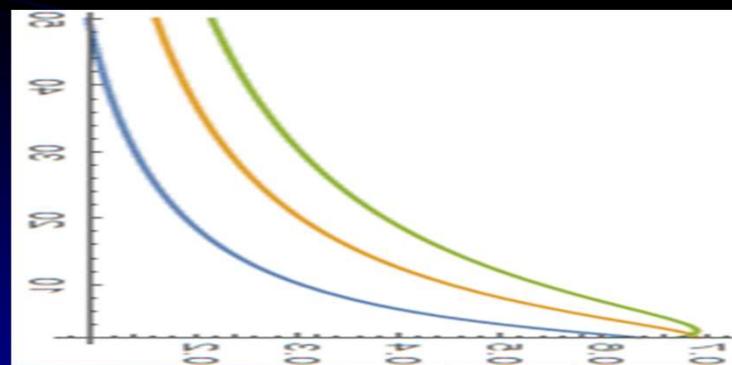
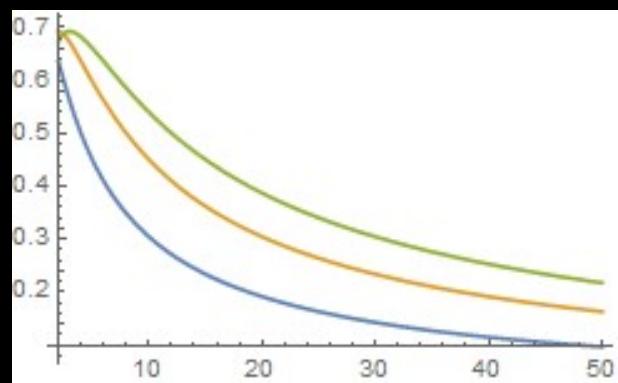
Shannon Max

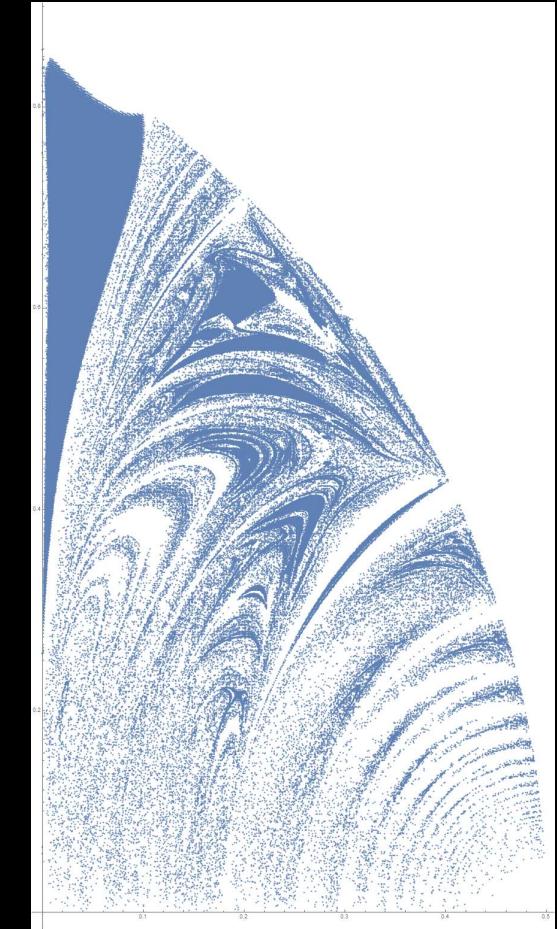
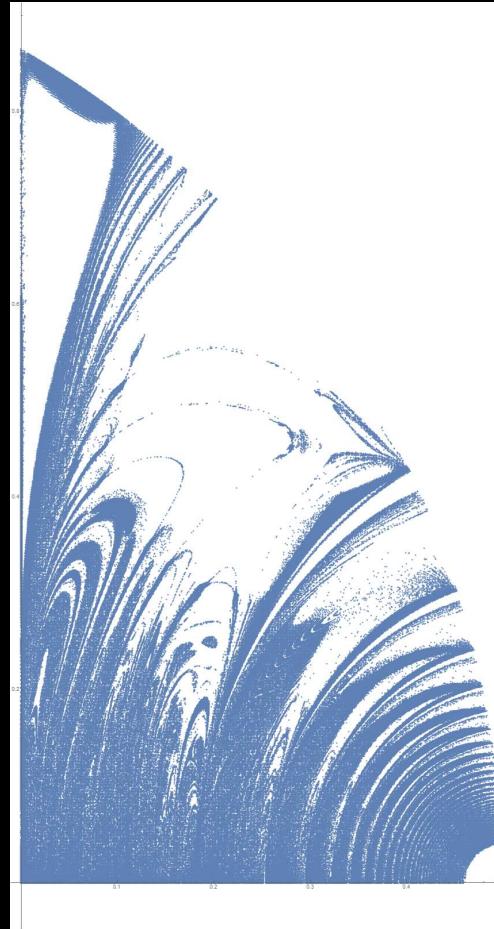
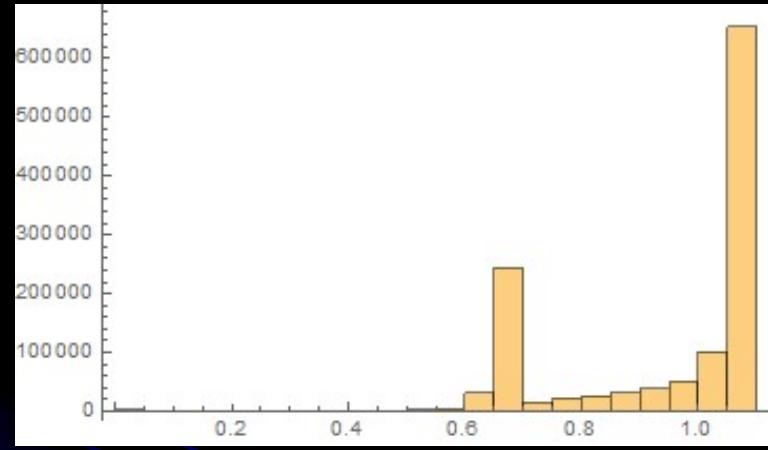


$$f[x_, y_] = -x \operatorname{Log}[x / (y + x)] / (y + x) - y \operatorname{Log}[y / (y + x)] / (y + x)$$

$$= \frac{x \operatorname{Log}\left[\frac{x}{x+y}\right]}{x+y} - \frac{y \operatorname{Log}\left[\frac{y}{x+y}\right]}{x+y}$$

```
Plot[{f[x, 1.], f[x, 2.], f[x, 3.]}, {x, 2., 50.}]
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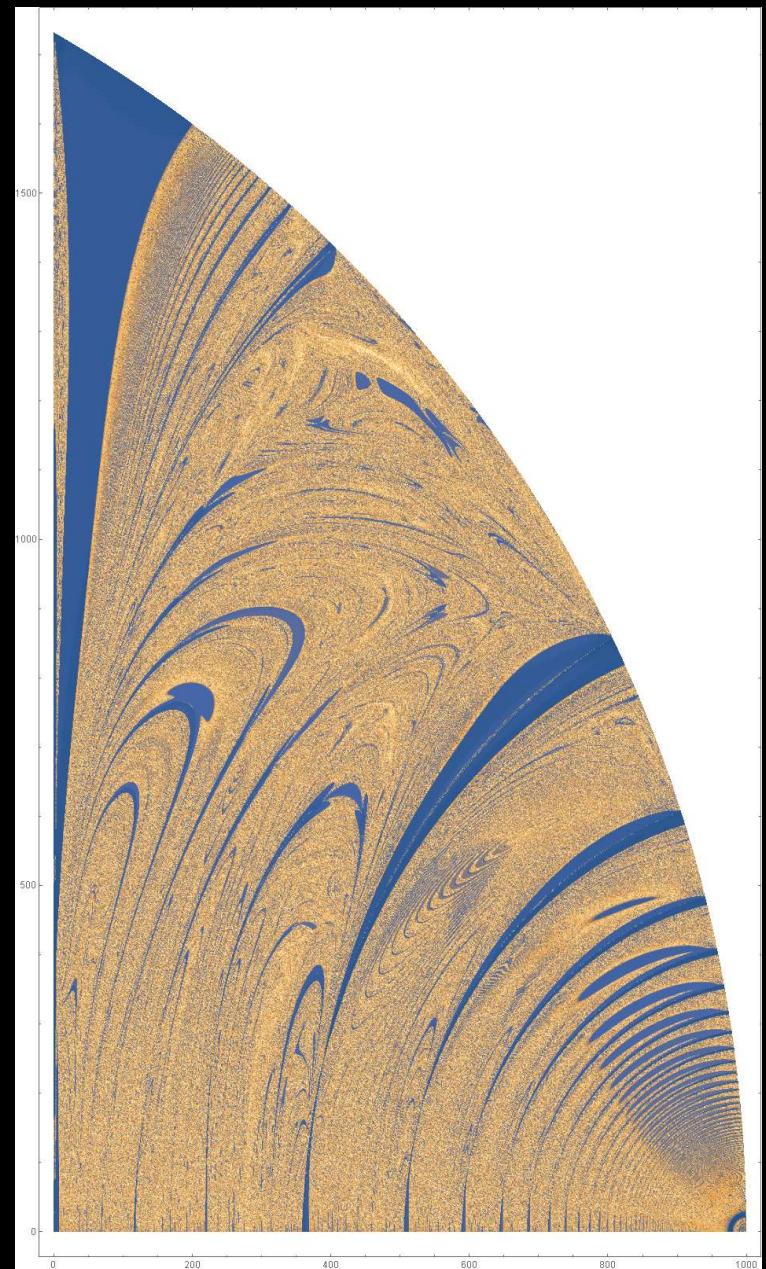
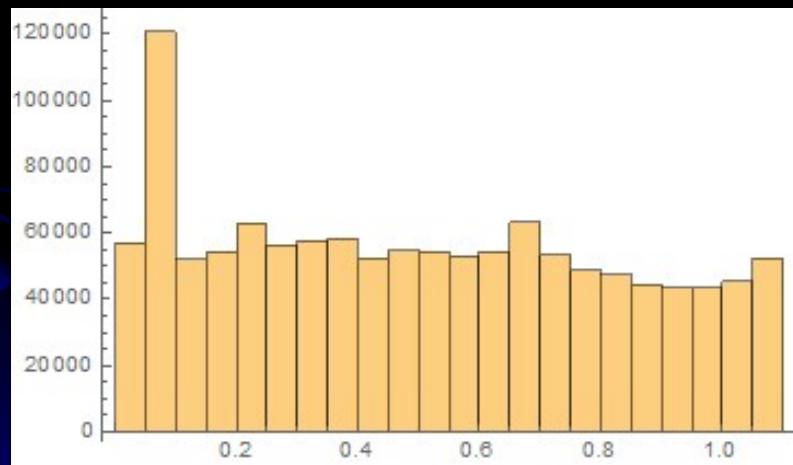




Initial conditions corresponding to
the two modes on the histogram

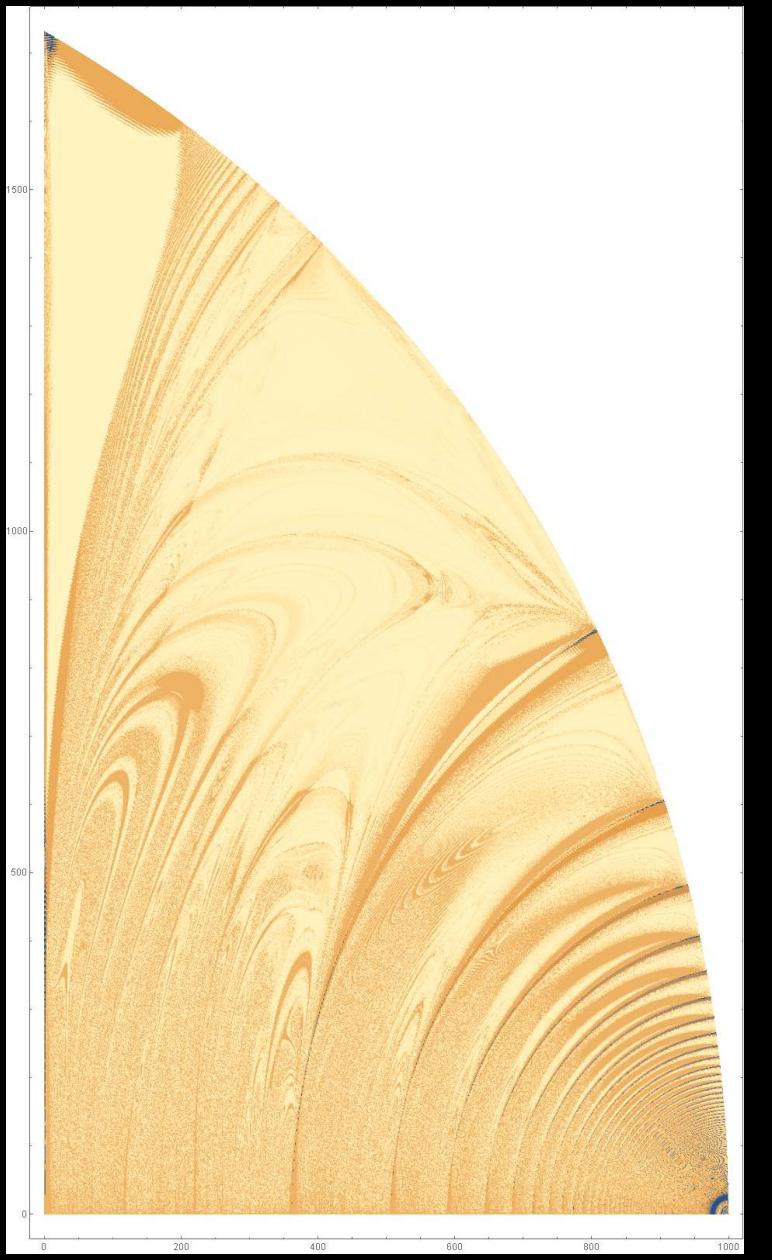
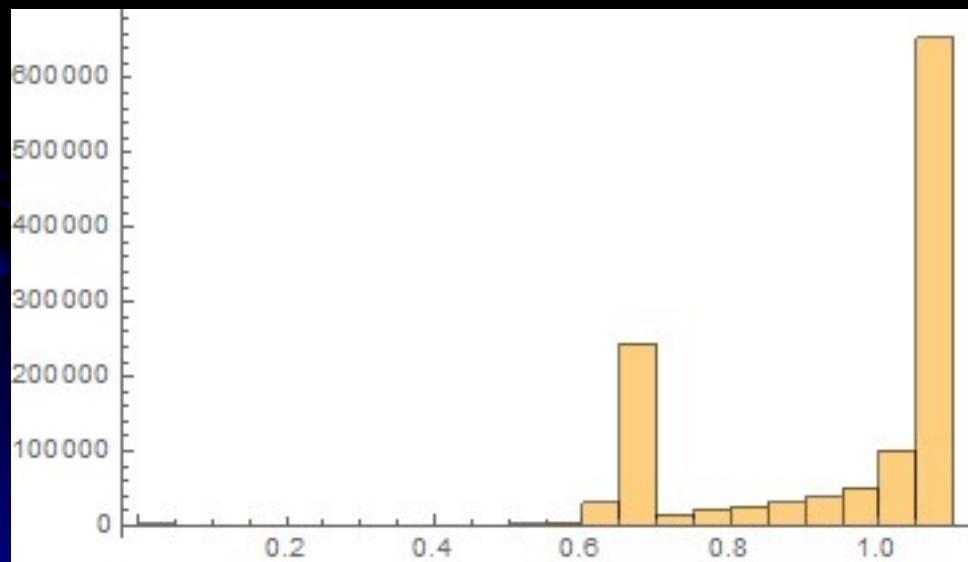
Shannon (full) entropy

49



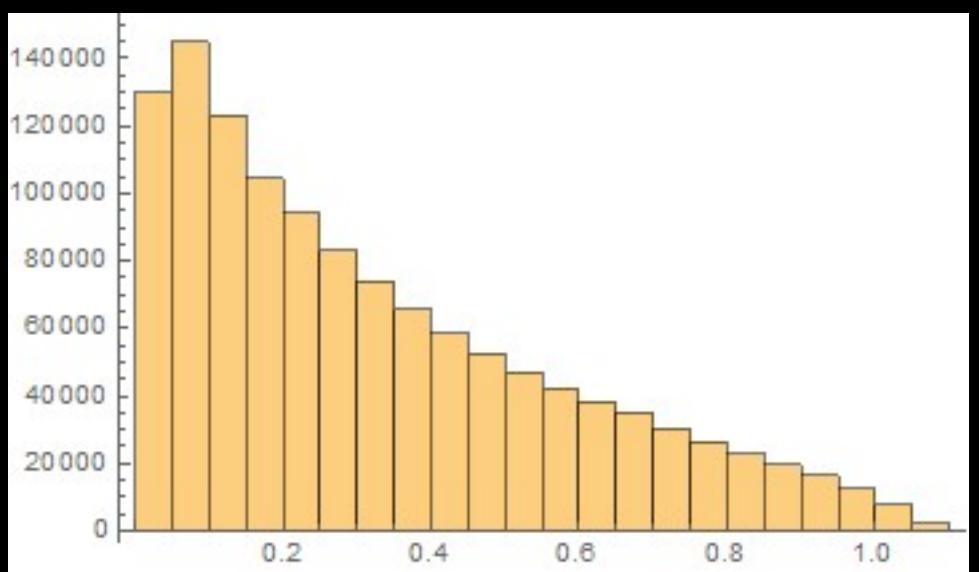
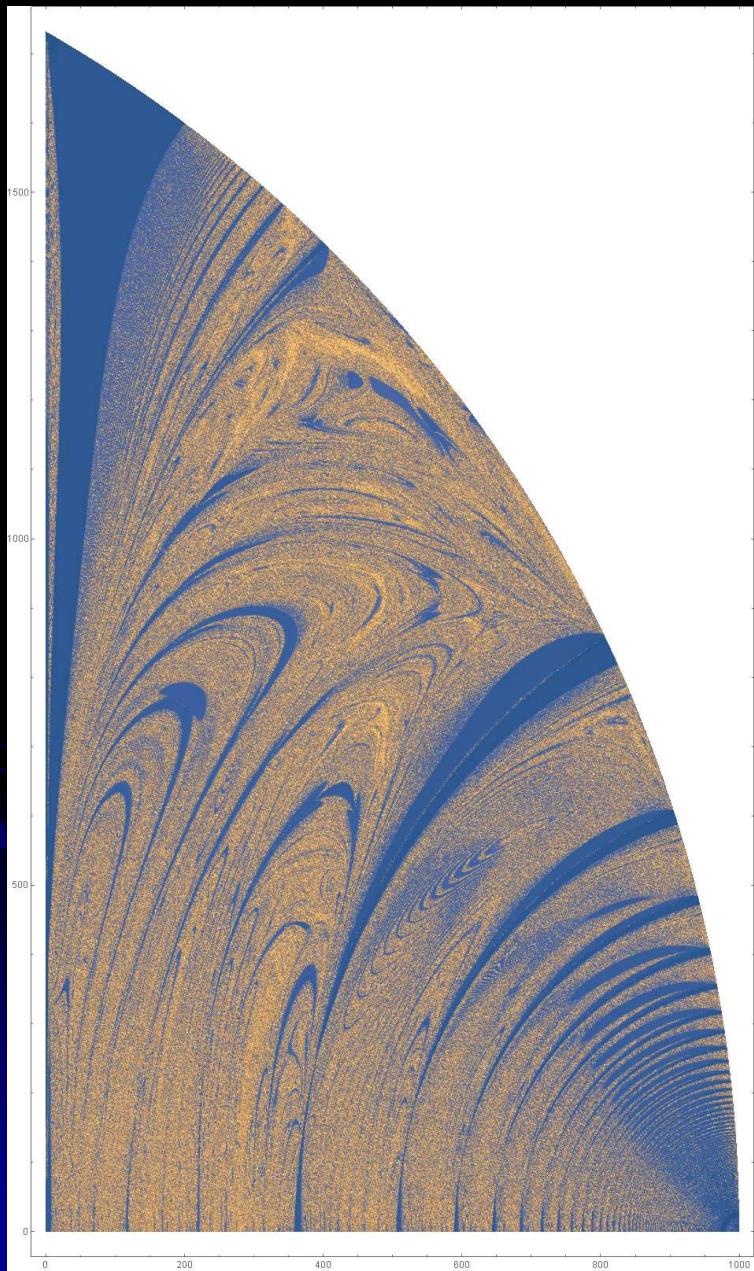
max entropy

192685



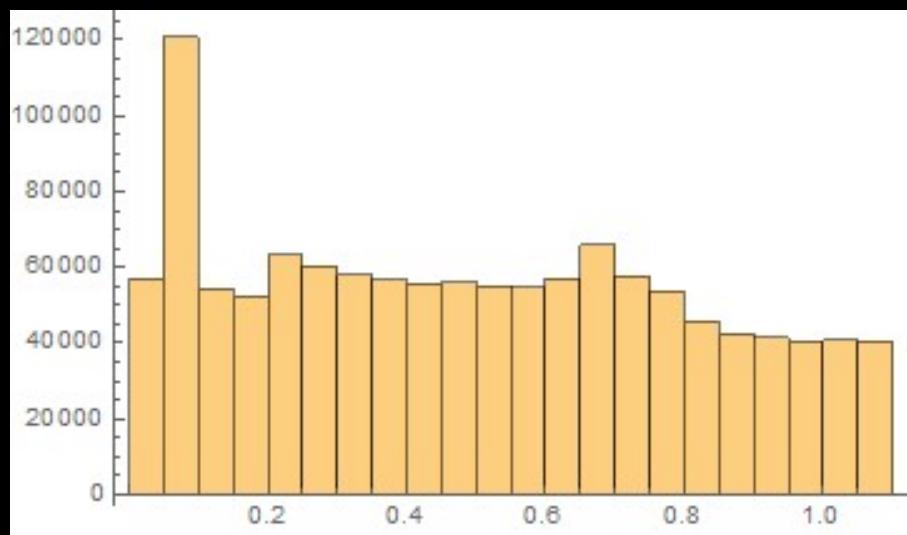
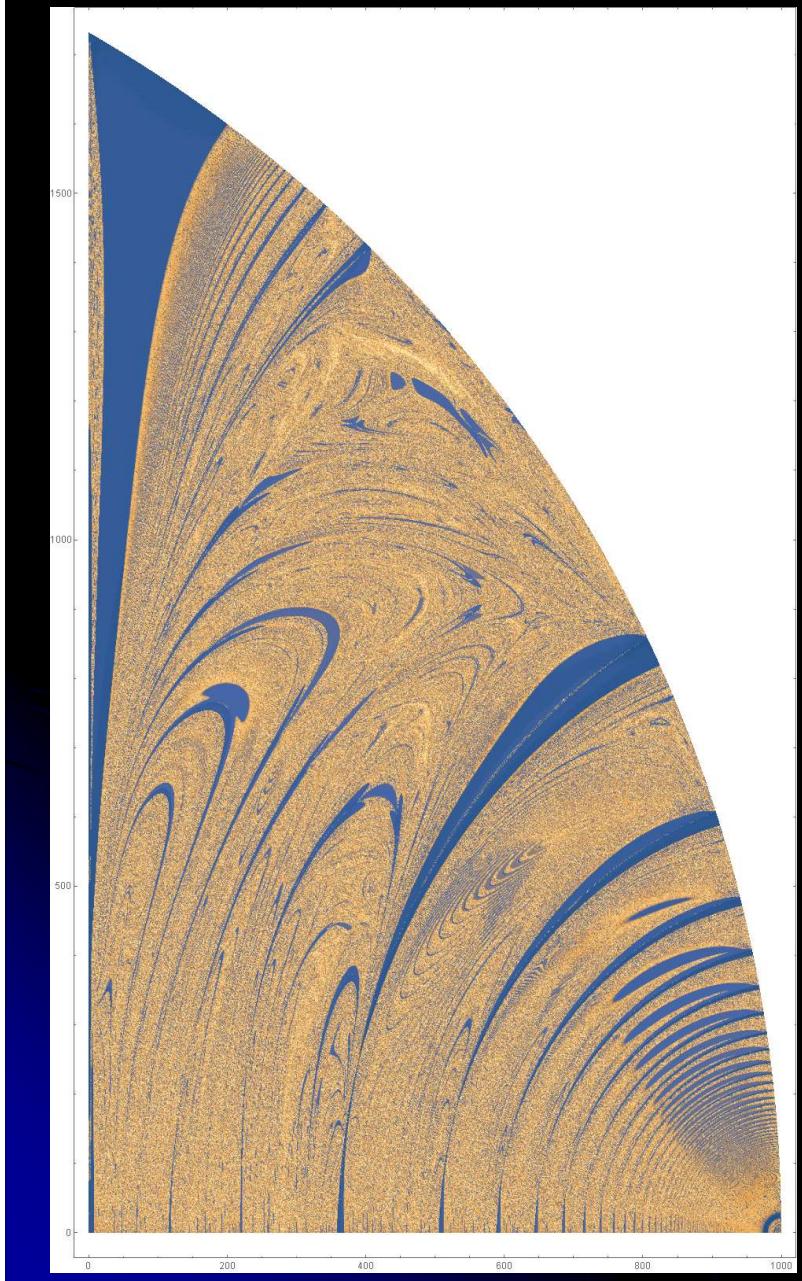
M¹

1



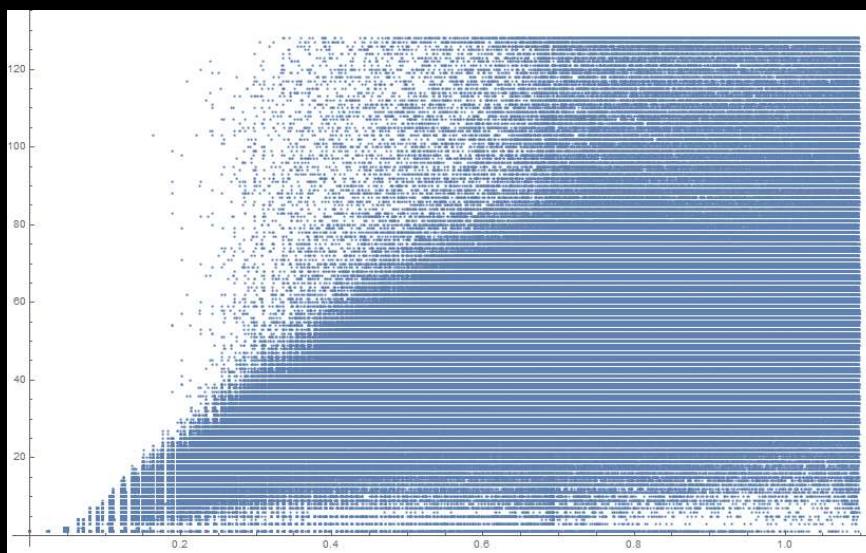
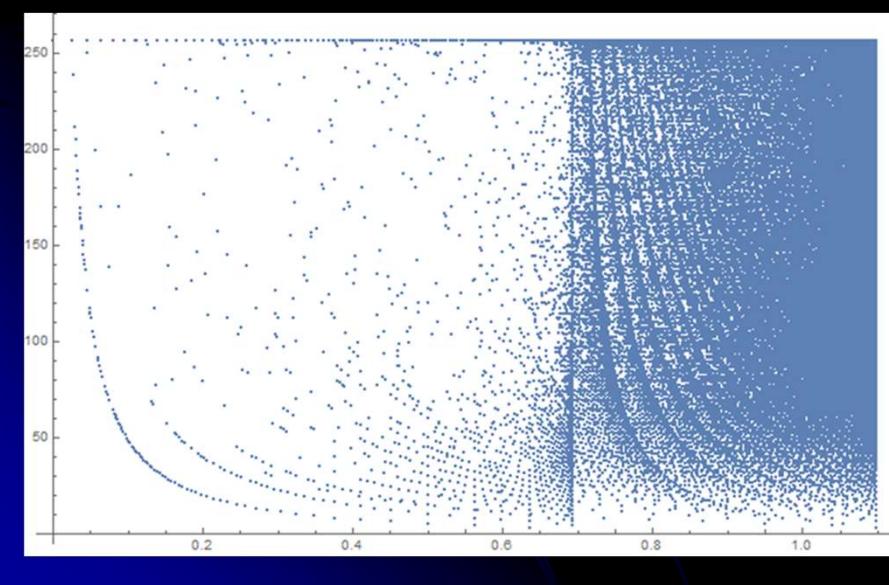
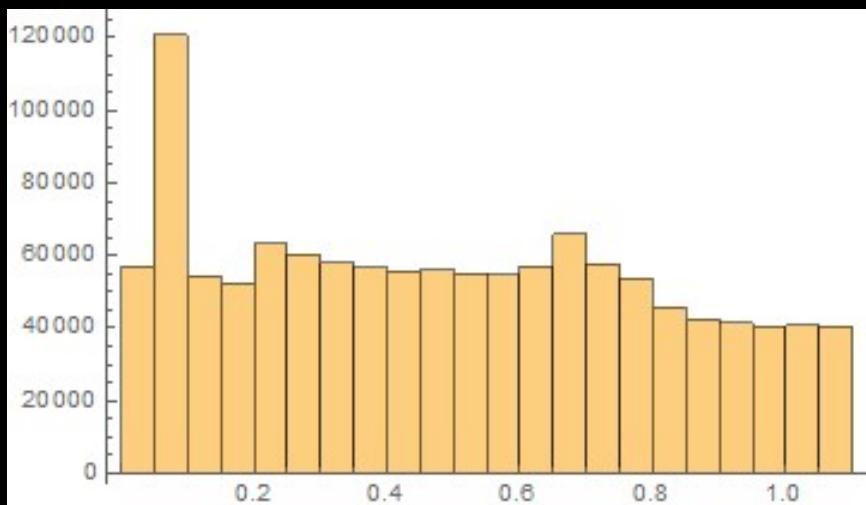
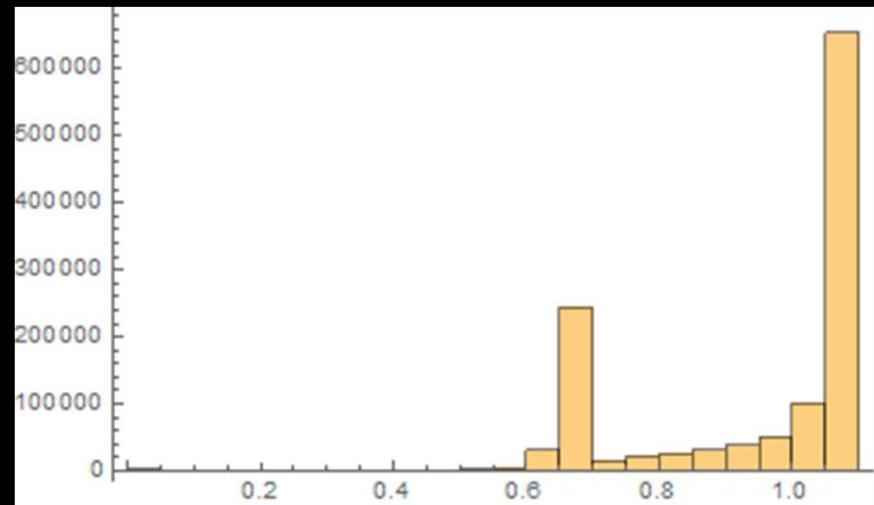
M_{\max}

1



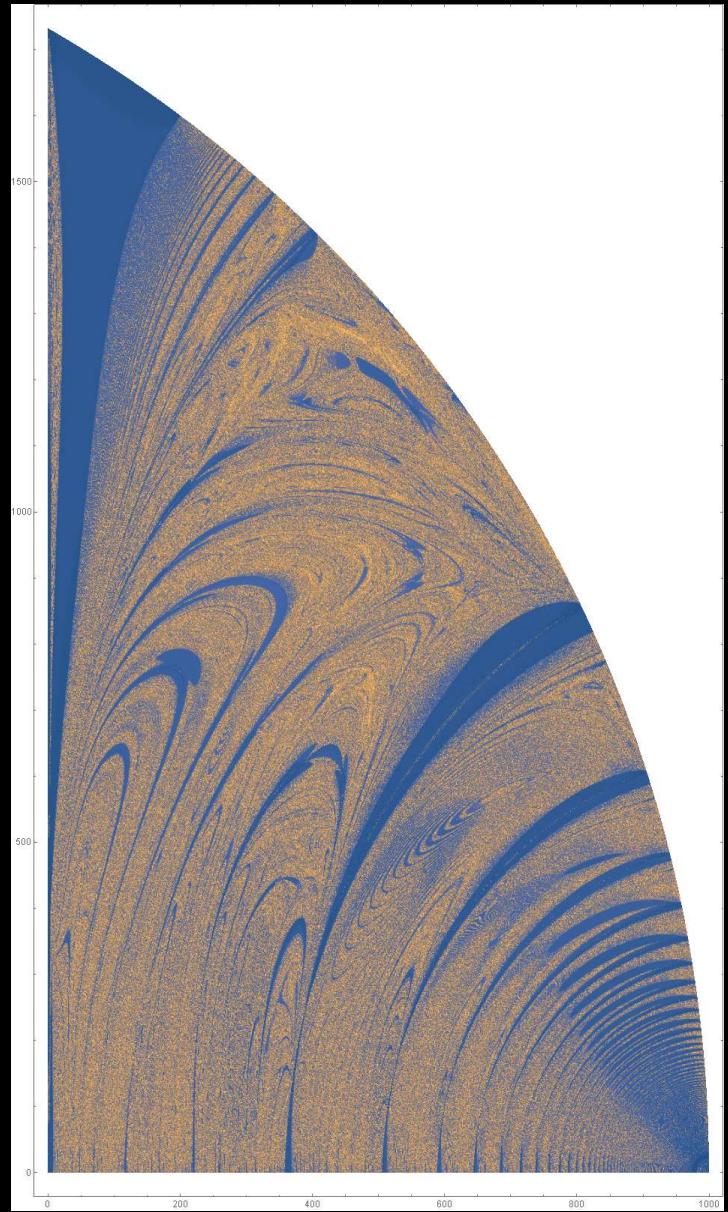
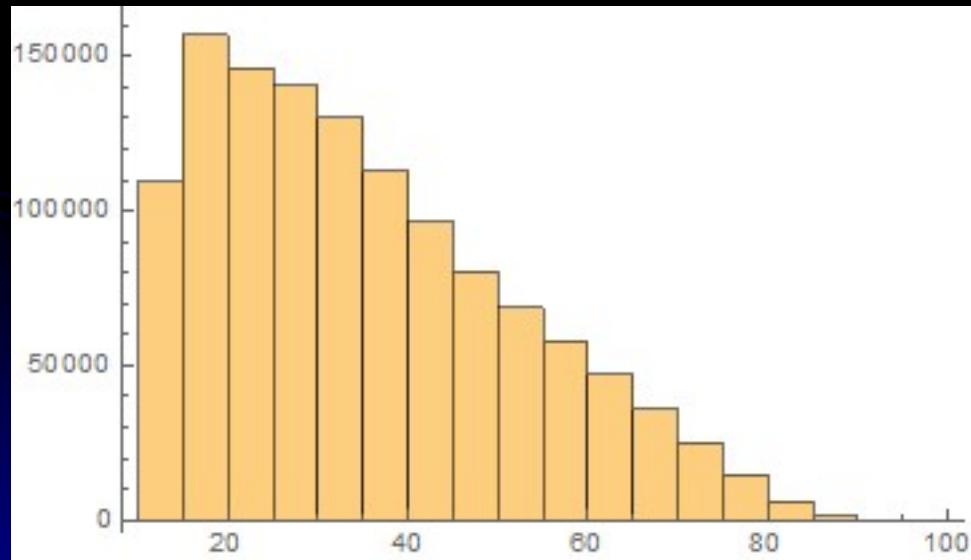
Sh_{max}

M_{max}



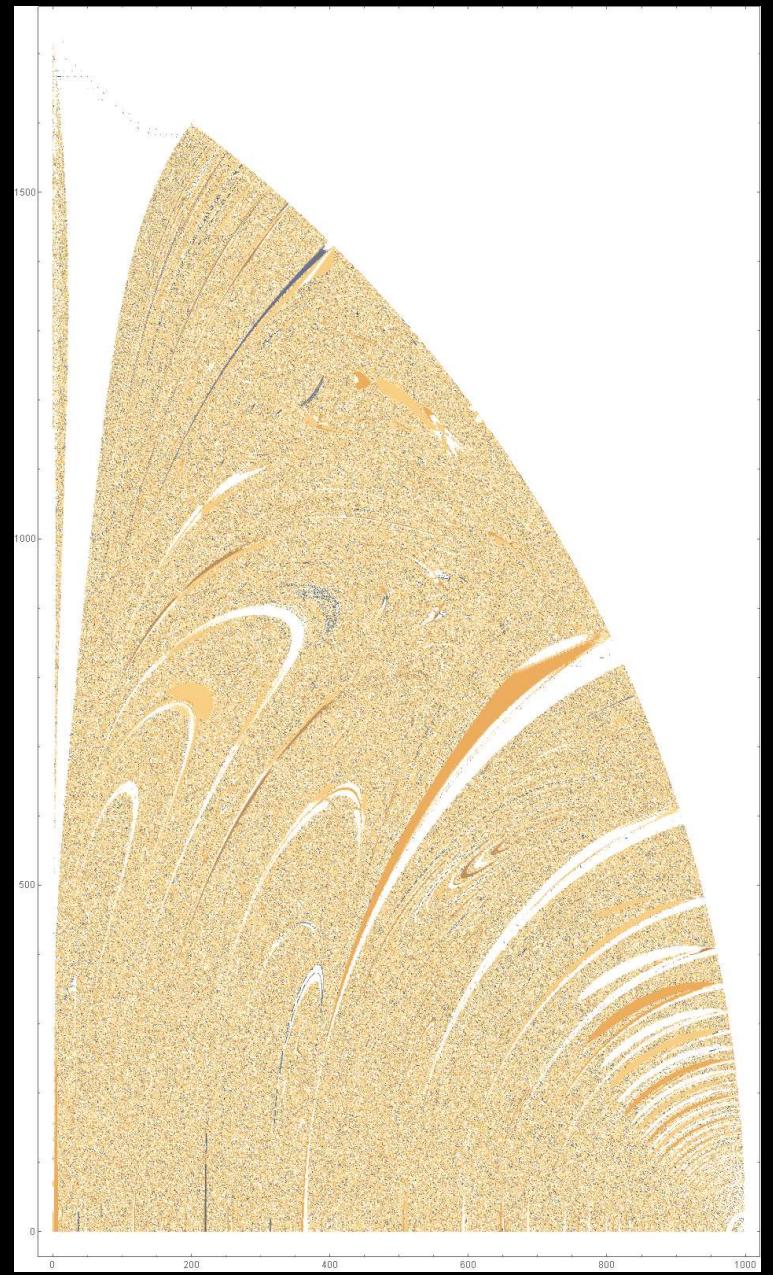
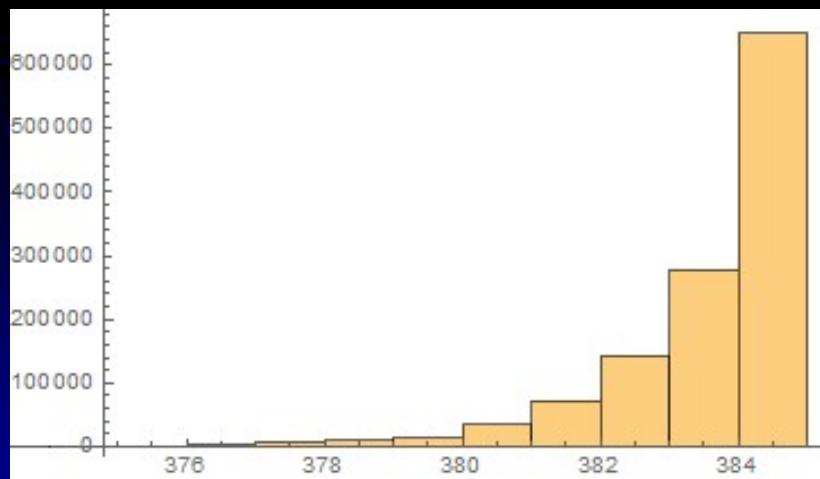
Kolmogorov complexity

1



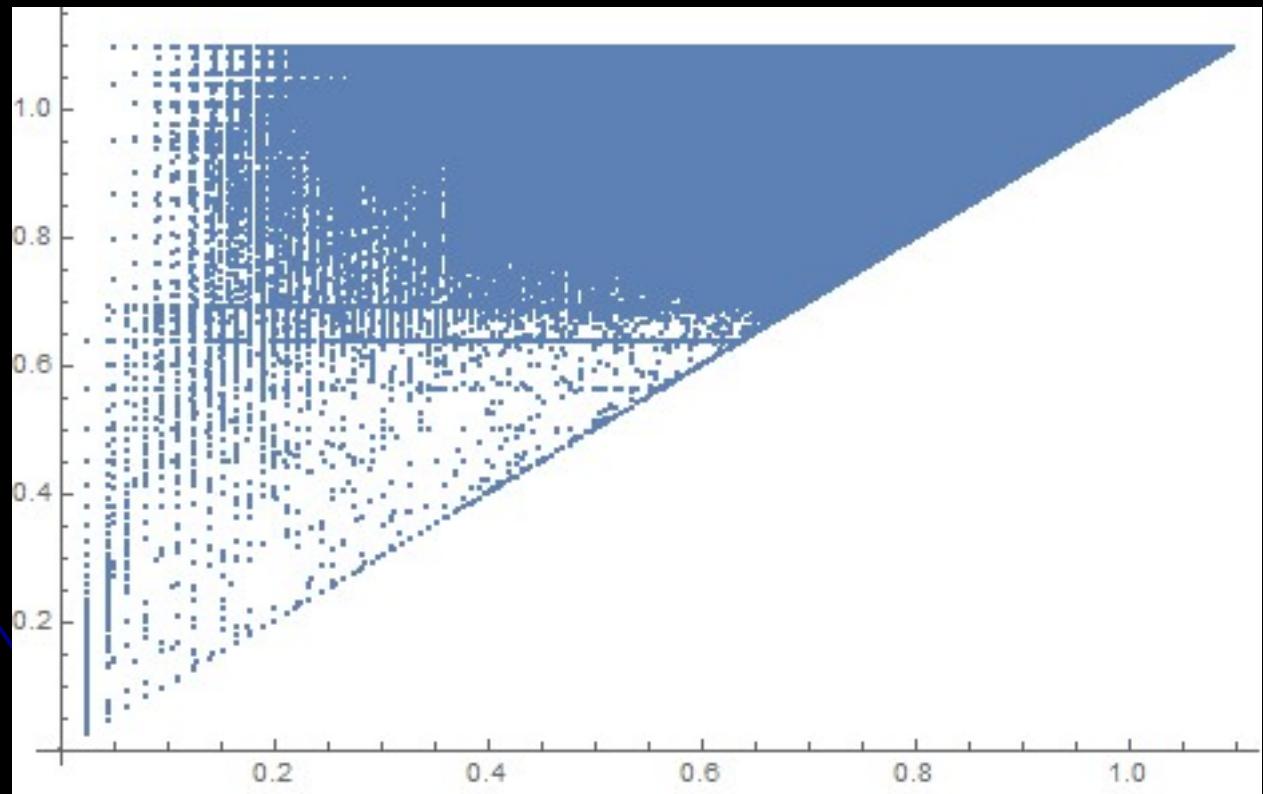
A₂₅₆

649641



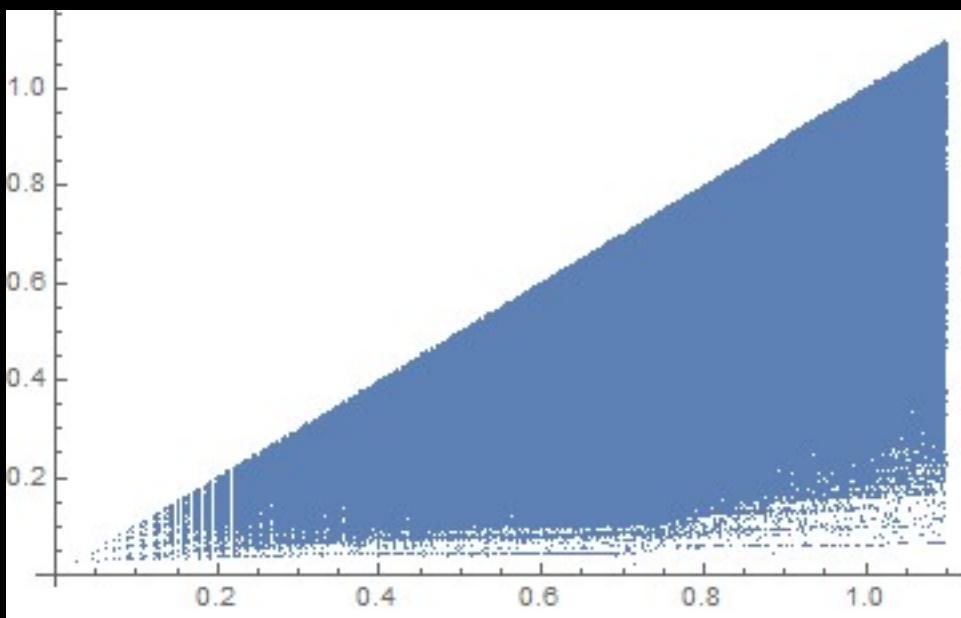
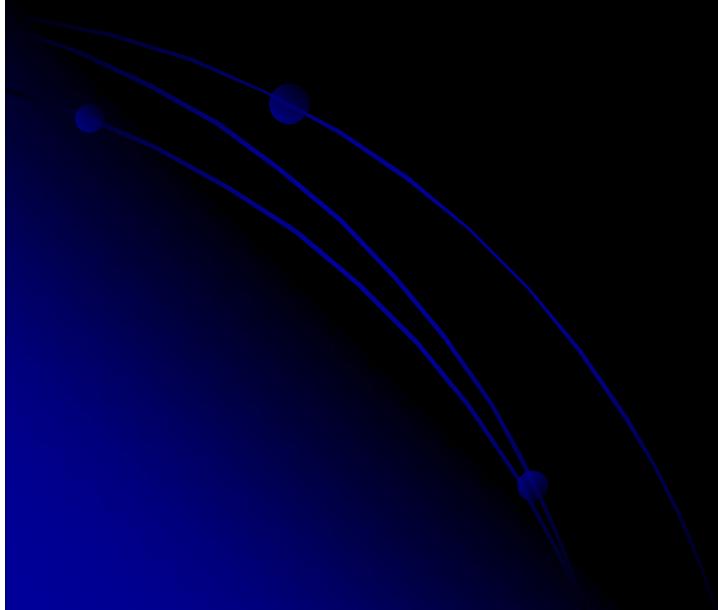
Sh - Sh_{Max}

$$\begin{pmatrix} 1. & 0.446756 \\ 0.446756 & 1. \end{pmatrix}$$



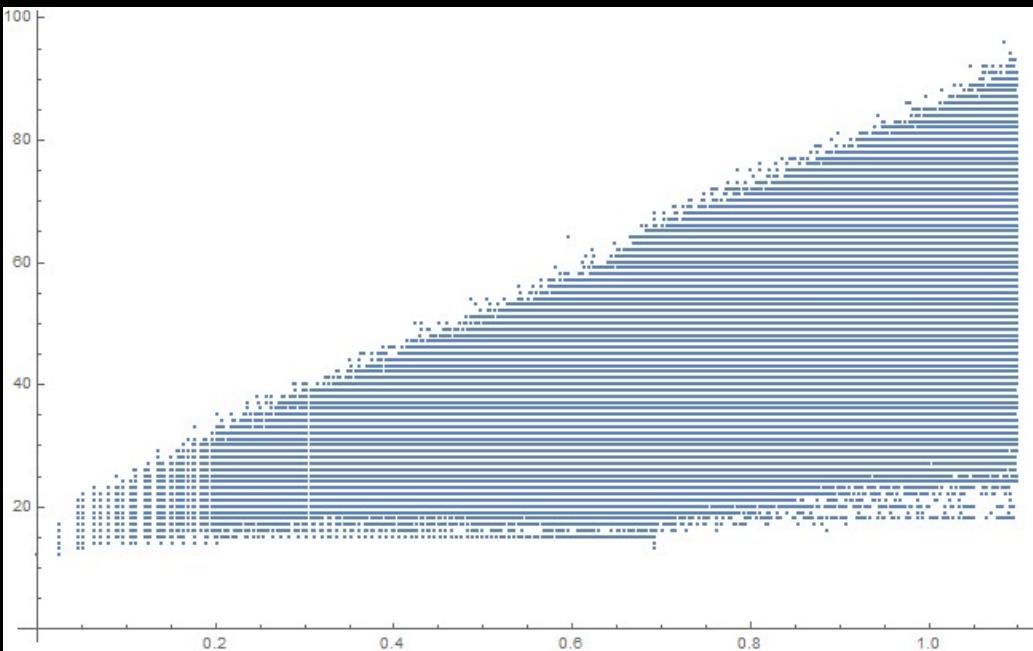
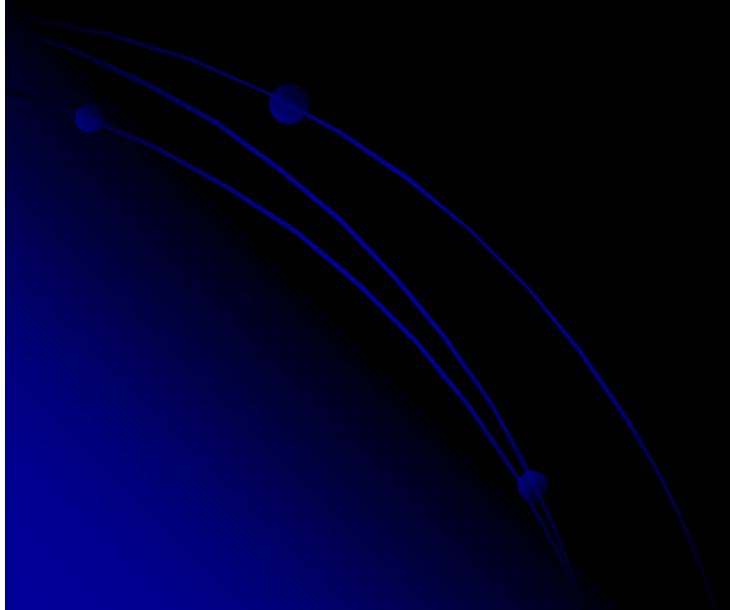
Sh - Mk¹

$$\begin{pmatrix} 1. & 0.857774 \\ 0.857774 & 1. \end{pmatrix}$$



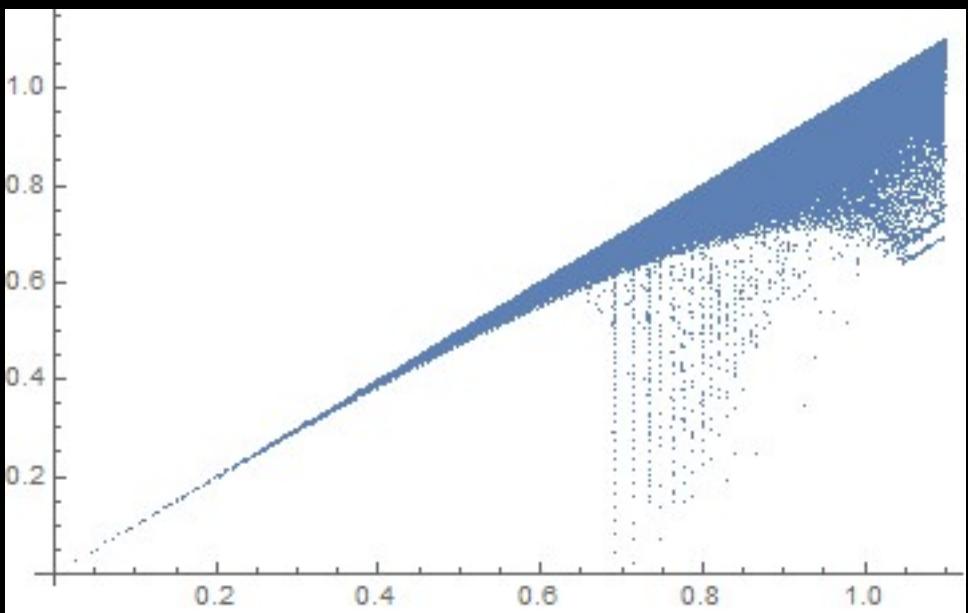
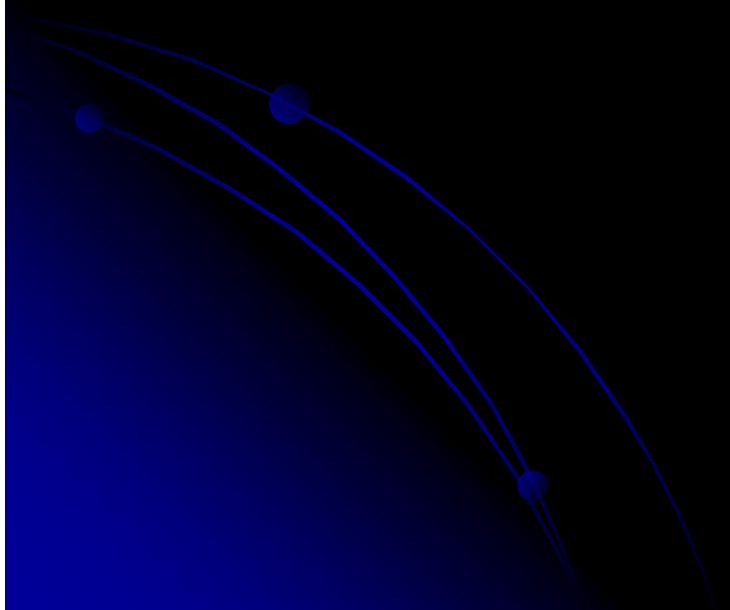
Sh - K

$$\begin{pmatrix} 1. & 0.873823 \\ 0.873823 & 1. \end{pmatrix}$$



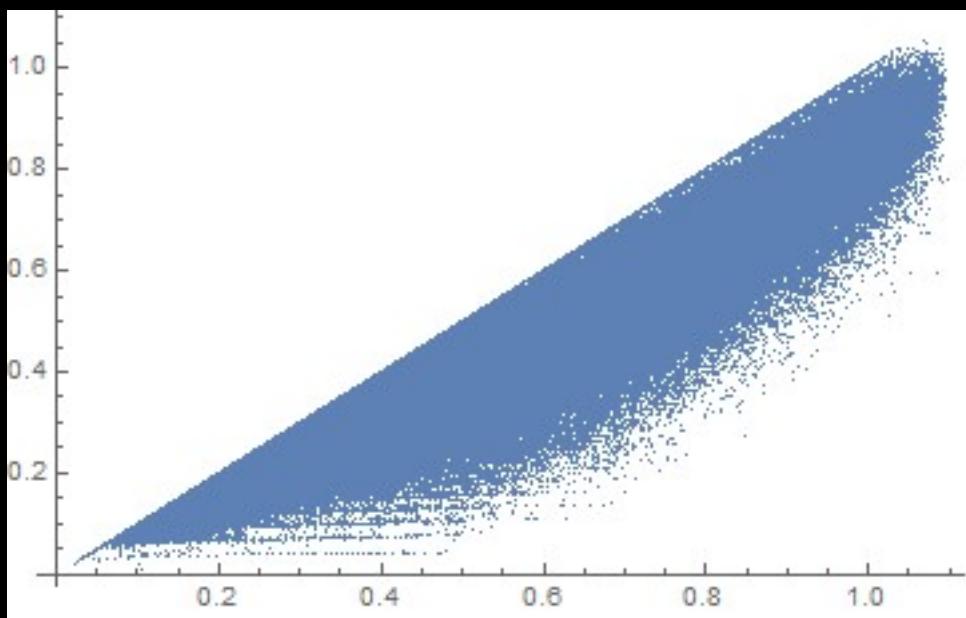
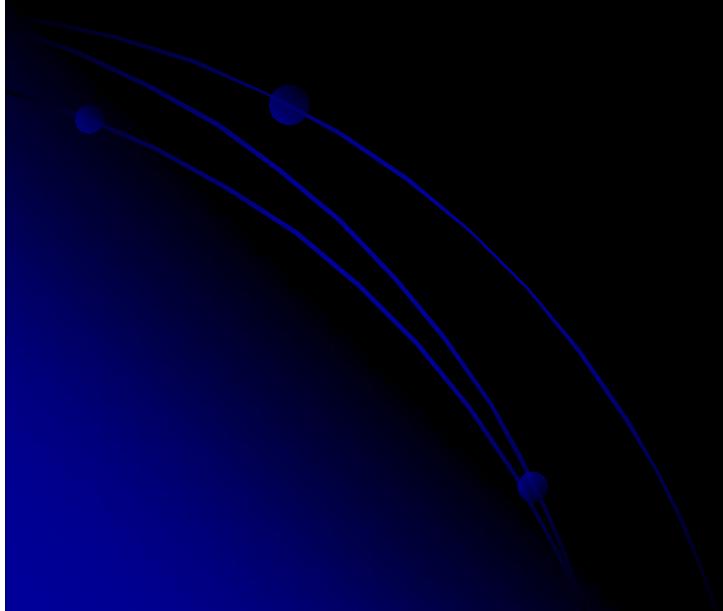
Sh - Mk_{Max}

$$\begin{pmatrix} 1. & 0.997759 \\ 0.997759 & 1. \end{pmatrix}$$



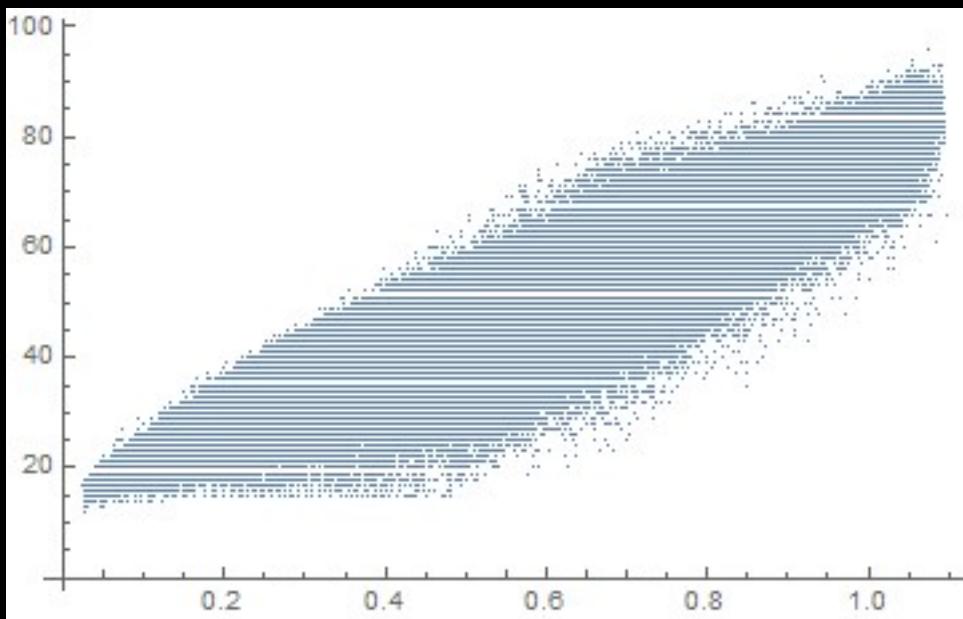
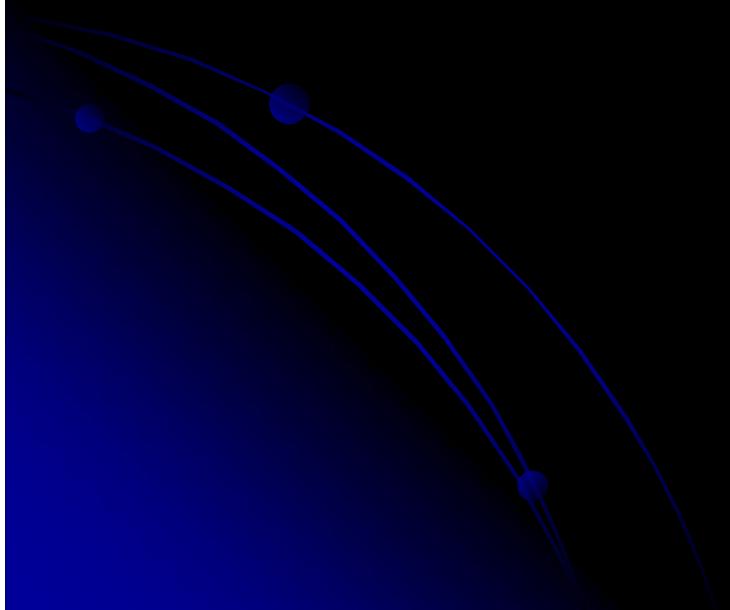
$$Mk^1 - Mk_{Min}$$

$$\begin{pmatrix} 1. & 0.979392 \\ 0.979392 & 1. \end{pmatrix}$$



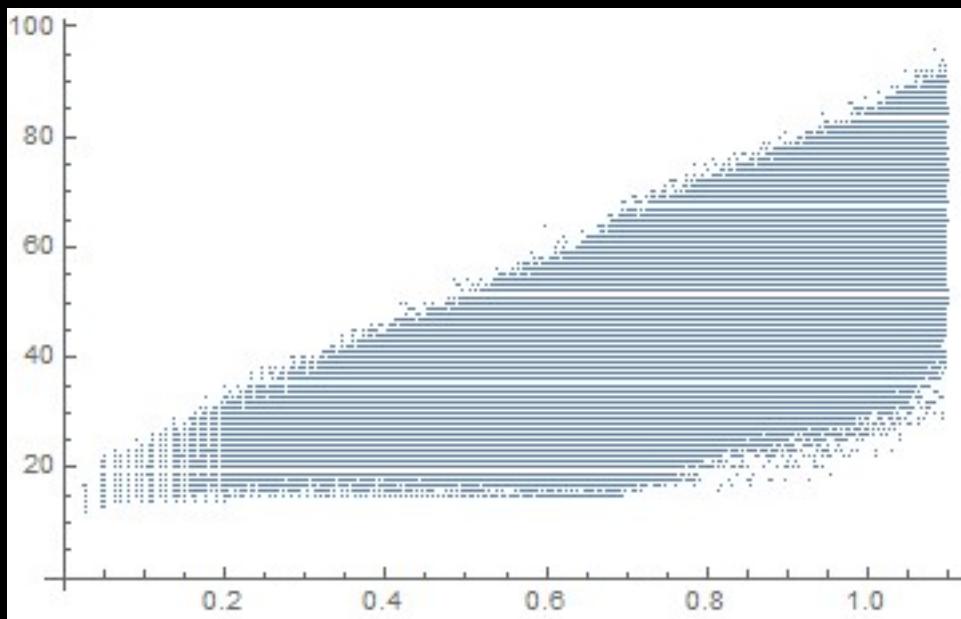
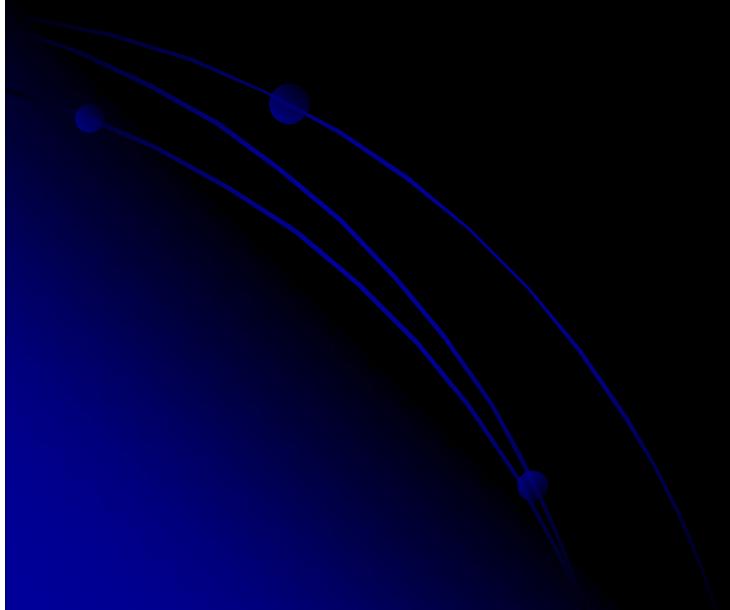
Mk¹ - K

$$\begin{pmatrix} 1. & 0.970332 \\ 0.970332 & 1. \end{pmatrix}$$



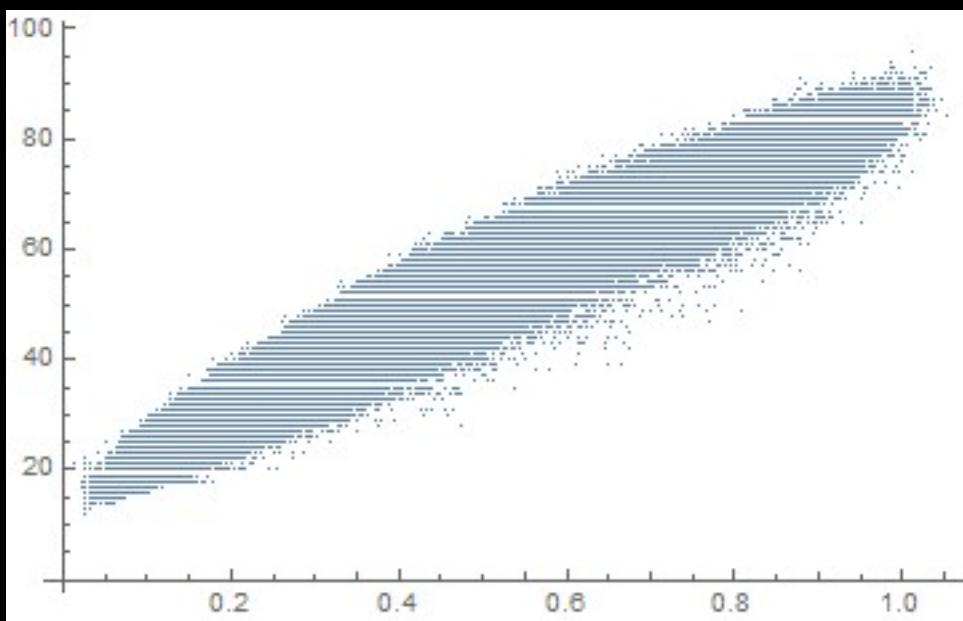
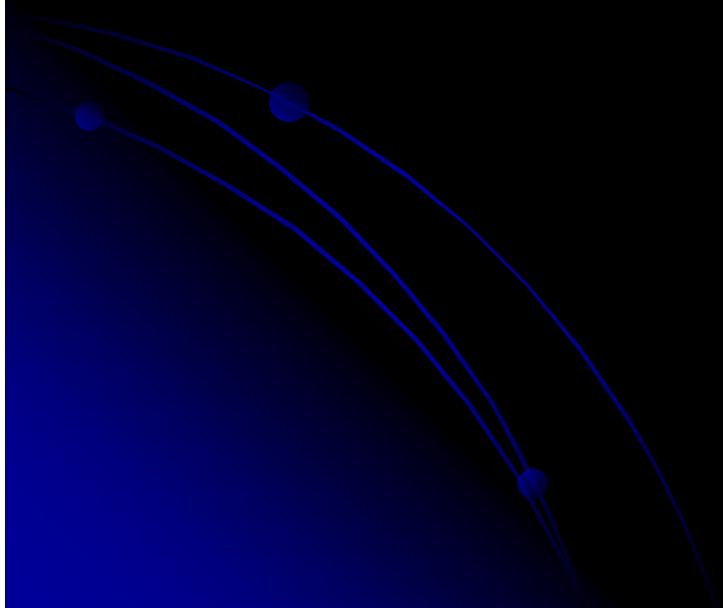
$Mk_{Max} - K$

$$\begin{pmatrix} 1. & 0.886194 \\ 0.886194 & 1. \end{pmatrix}$$



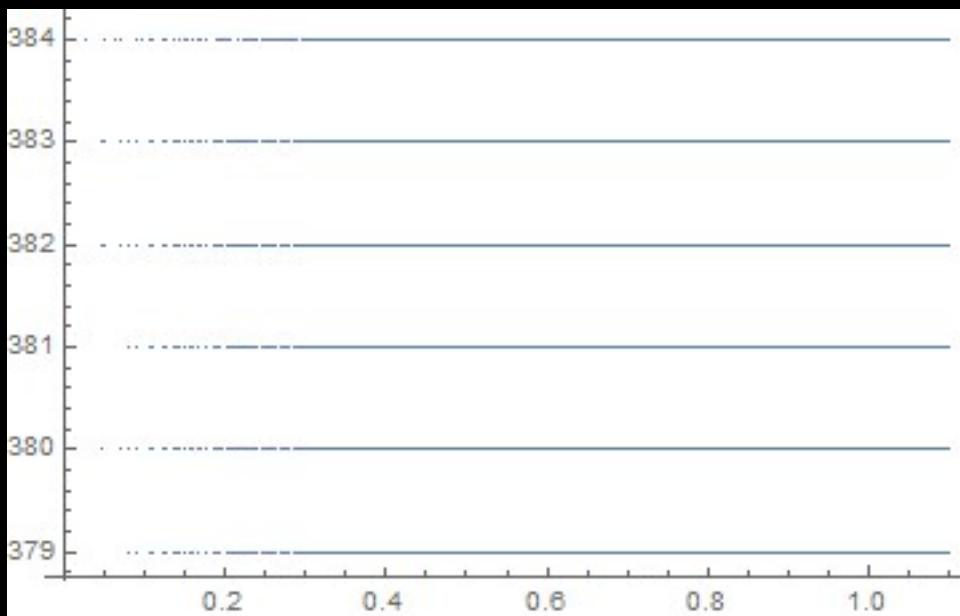
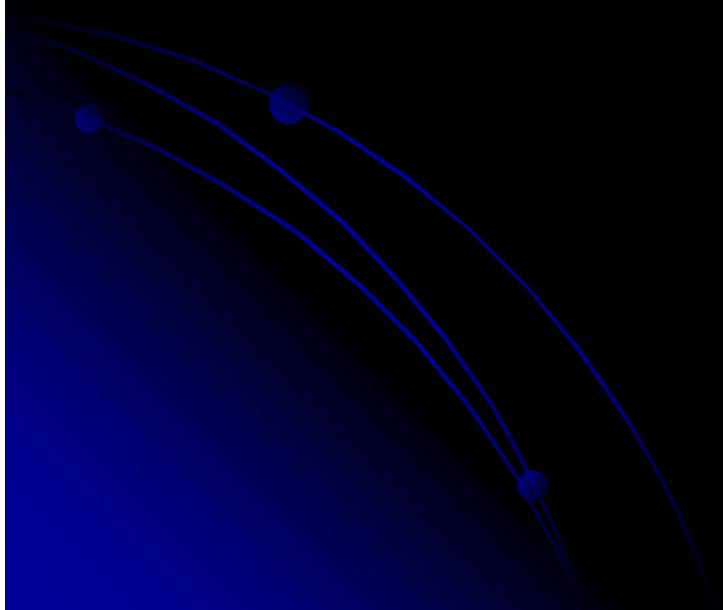
$Mk_{Min} - K$

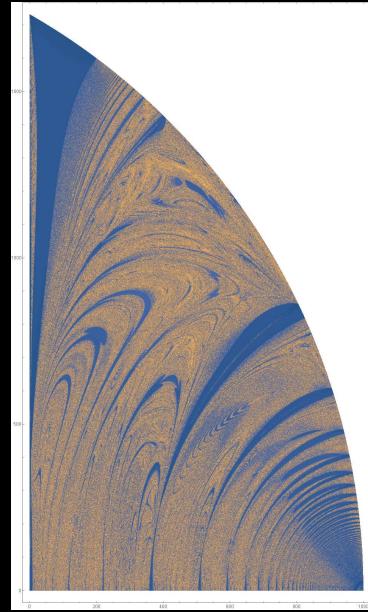
$$\begin{pmatrix} 1. & 0.983909 \\ 0.983909 & 1. \end{pmatrix}$$



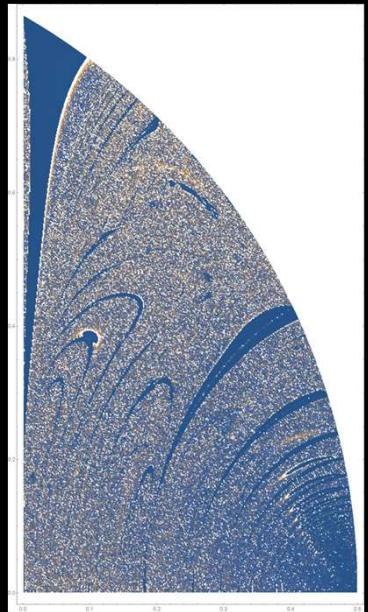
Sh - A₂₅₆

$$\begin{pmatrix} 1. & 0.140337 \\ 0.140337 & 1. \end{pmatrix}$$





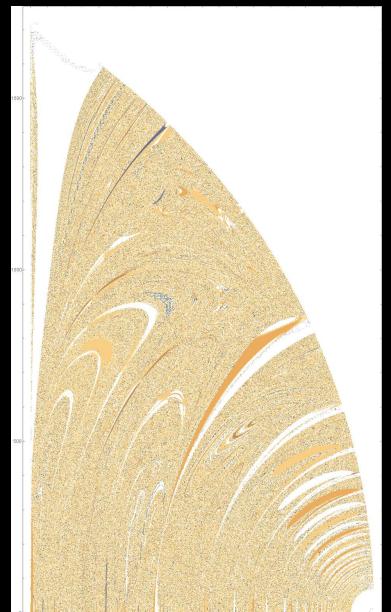
K



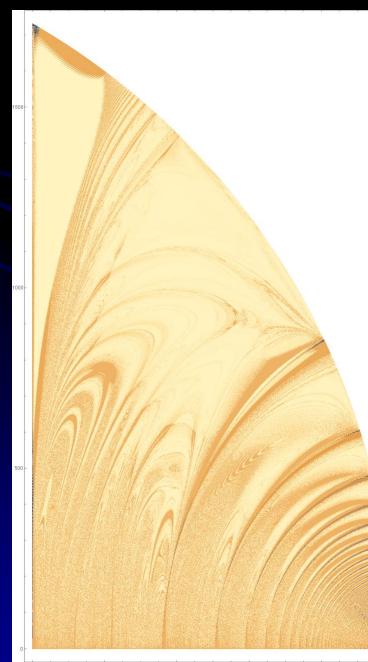
life-time



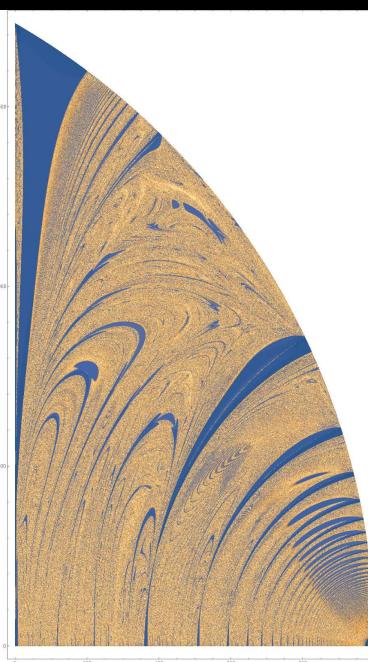
A_{256}



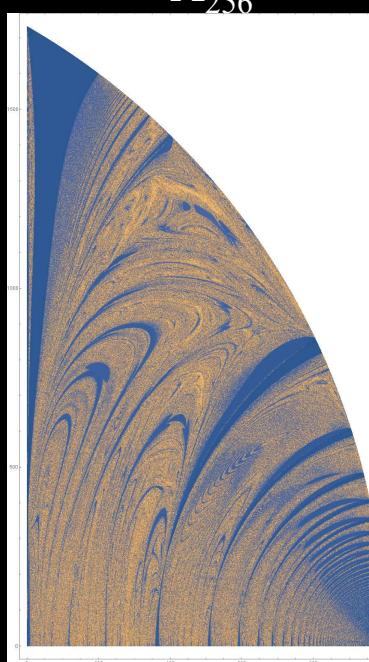
A_{64}



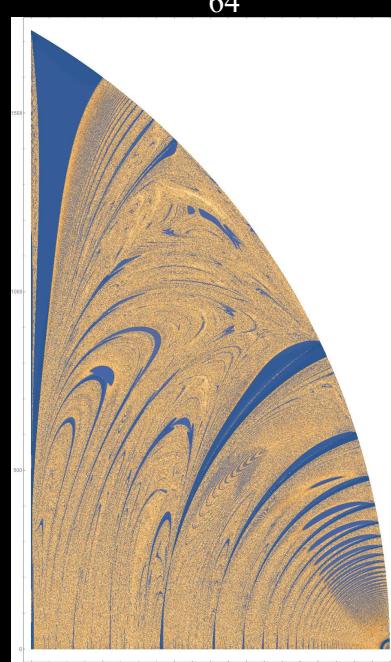
Sh_{max}



Sh



M^1



M_{max}

Arnold complexity

Consider a set M of all possible sequences of length n . Let us define (following Newton's idea) the increment sequence: we thus consider the linear operator $A: M \rightarrow M$,

$$y = Ax$$

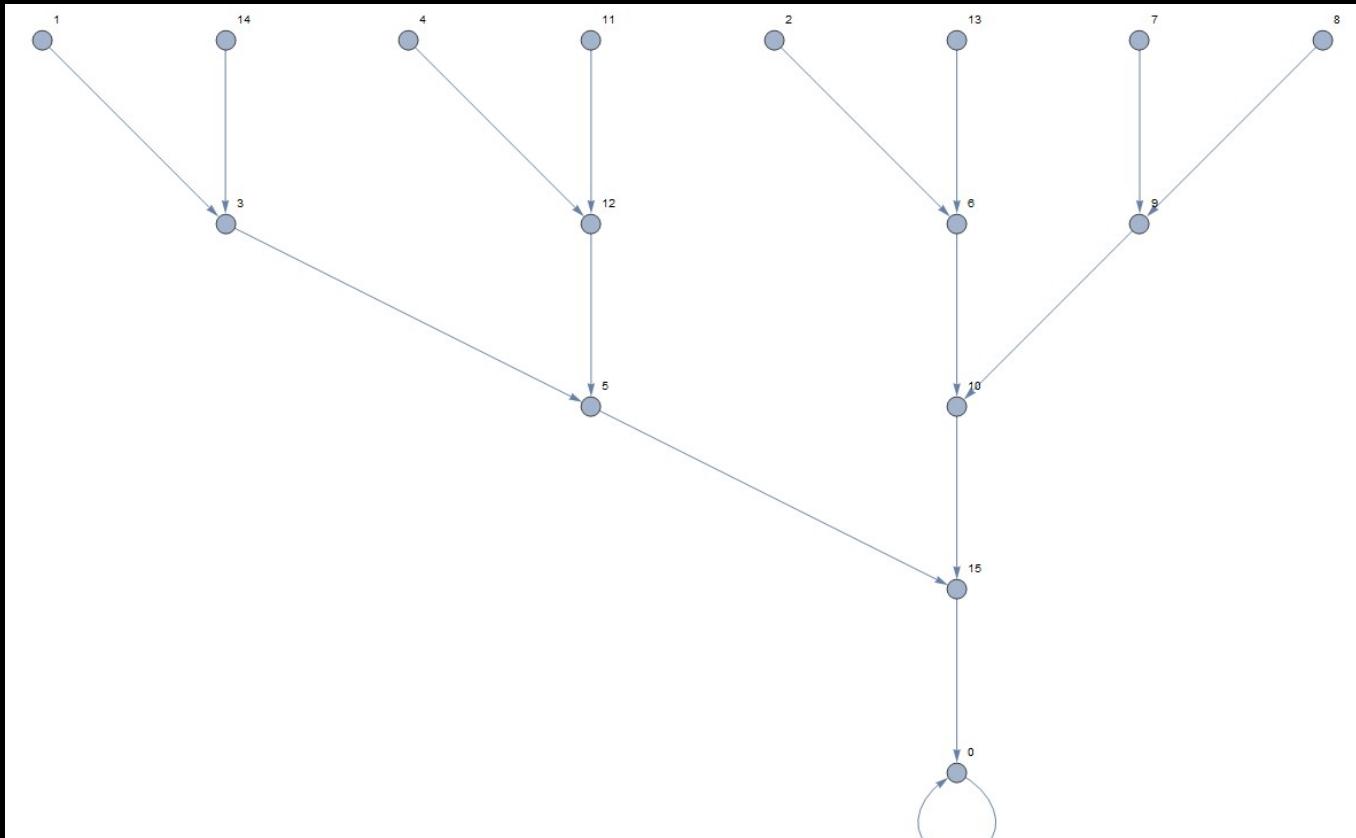
defined by the formula

$$y_j = x_{j+1} - x_j$$

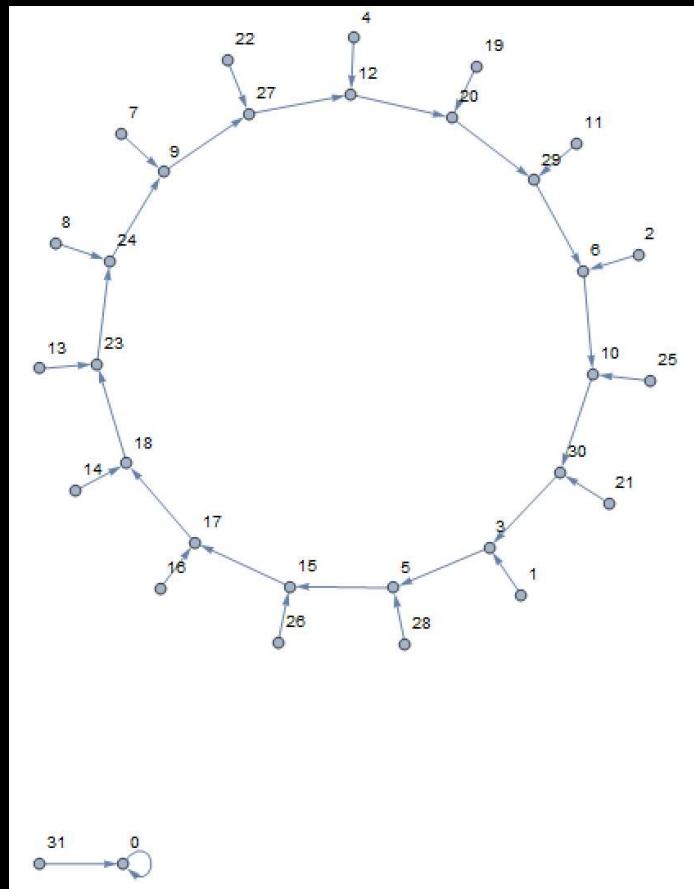
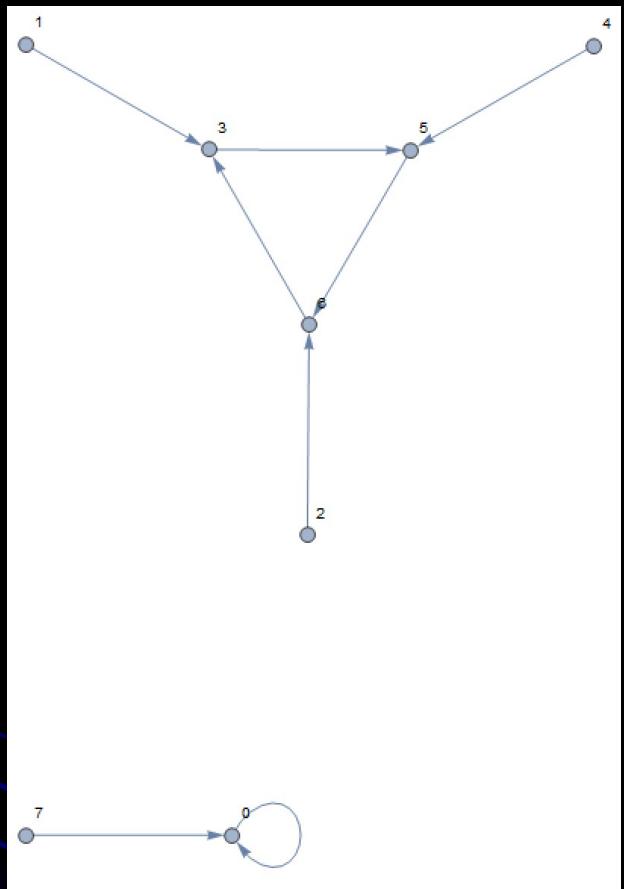
To have n increments, we define $x_{n+1} = x_1$, making our sequence x cyclic (the function x , whose value at j is x_j , is then n -periodic).

The definition of the complexity: we say that an object x is more complicated if the length of the cycle of the component of the graph containing the point x is larger. Inside the components whose cycles have equal lengths, a vertex is said to be more complicated if its distance from the cycle is larger.



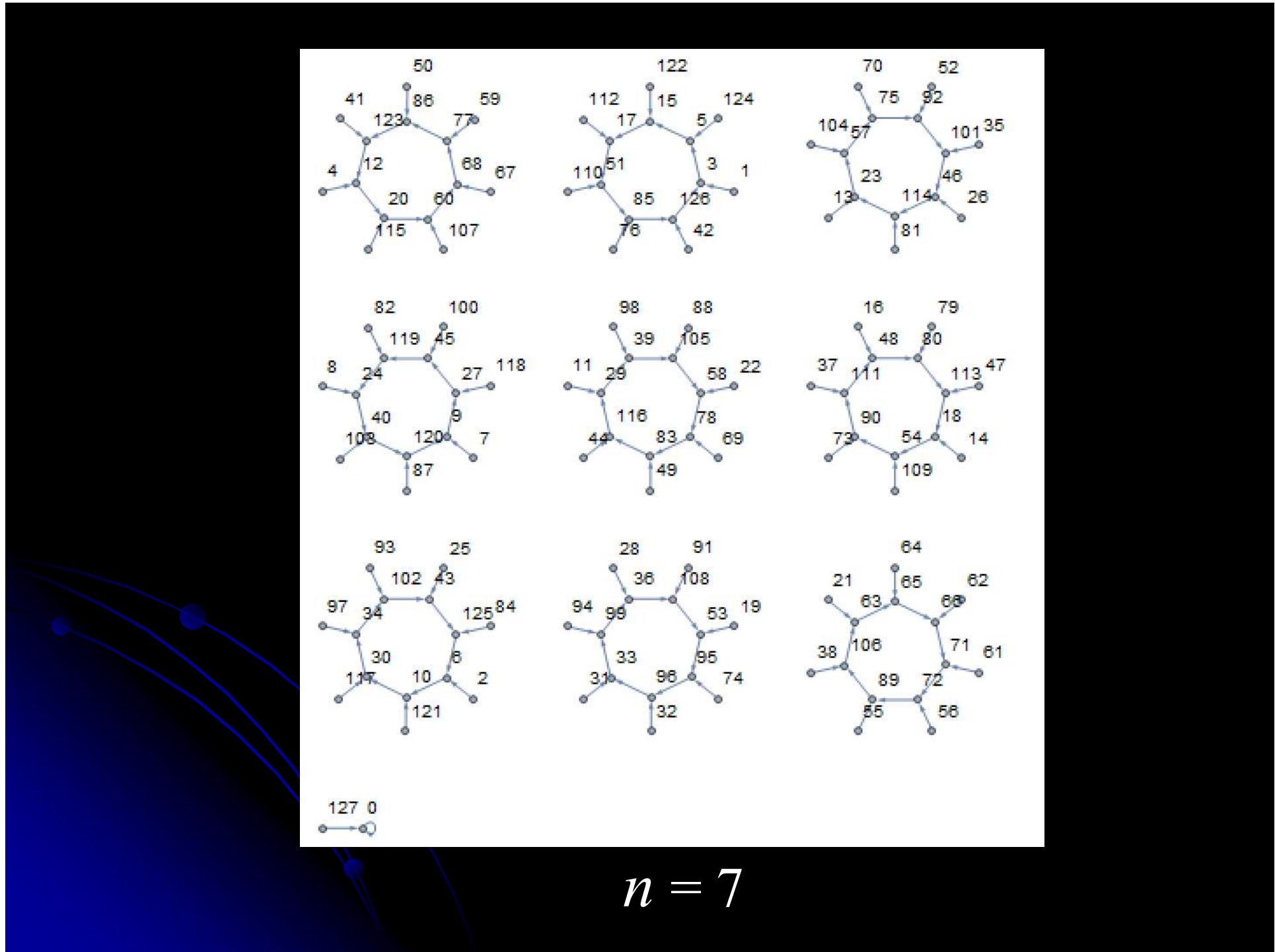


The map A of the finite set M into itself is described by a directed graph with 2^n vertices $x \in M$. In this graph, exactly one edge starts from each vertex x (and leads to Ax).

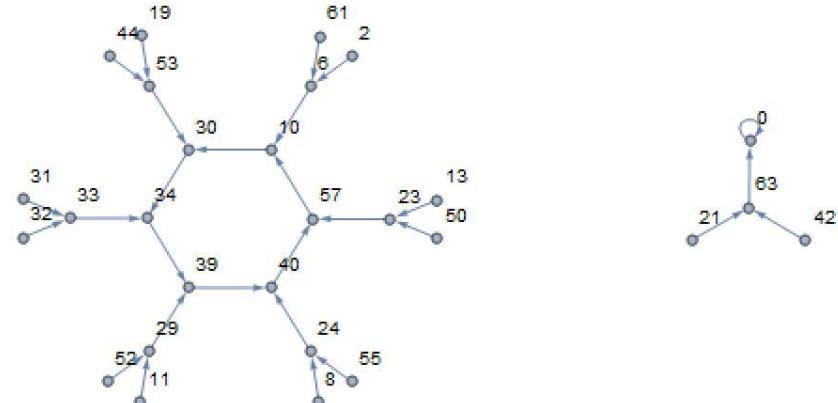
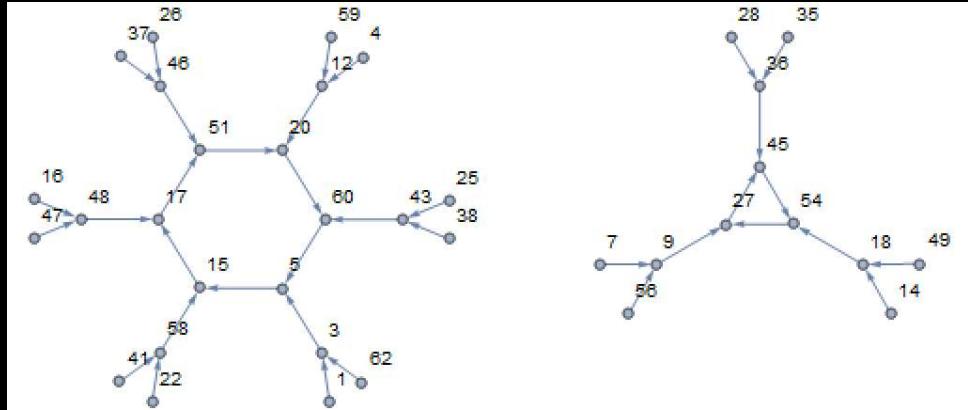


$n = 3$

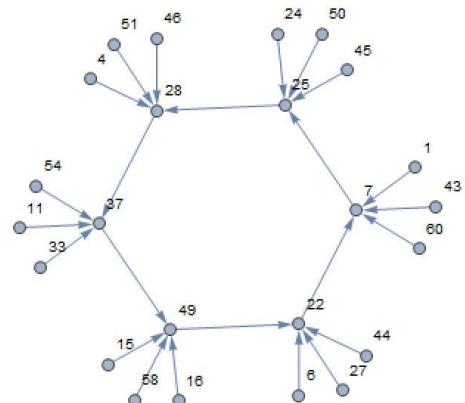
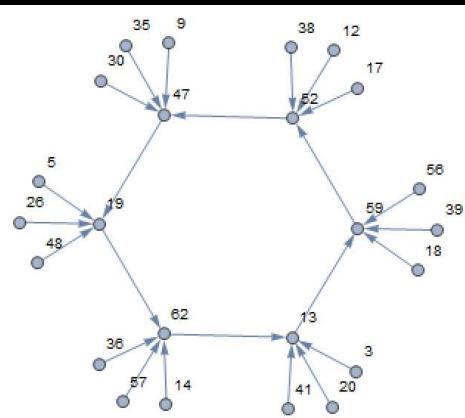
$n = 5$



binary



ternary



$n = 7$

$n = 3$

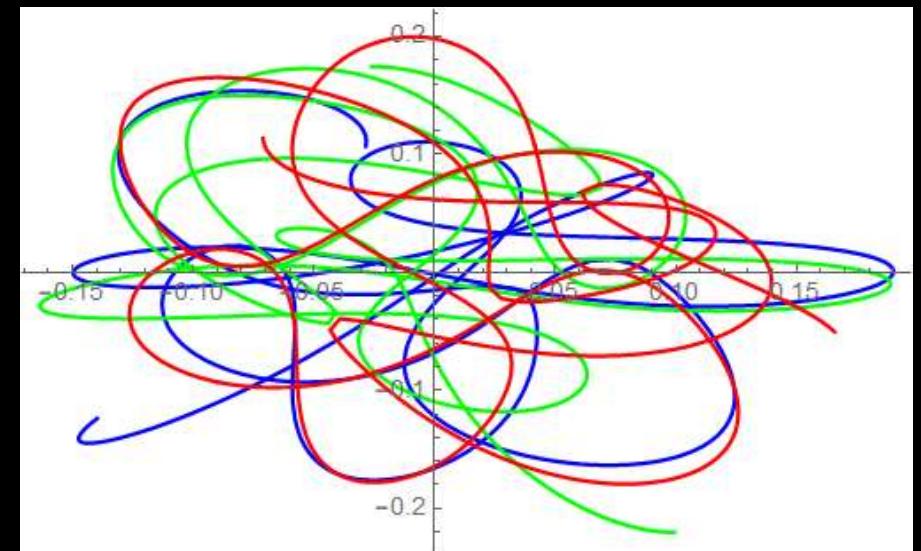
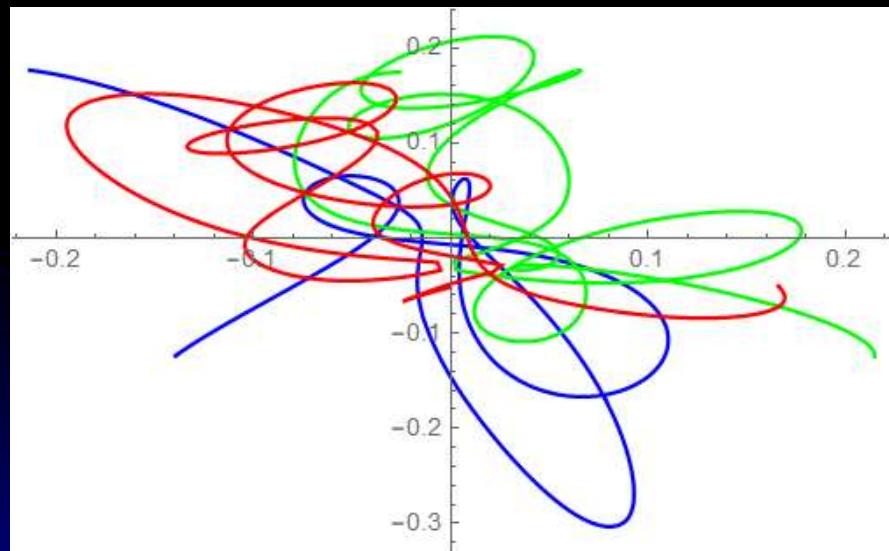
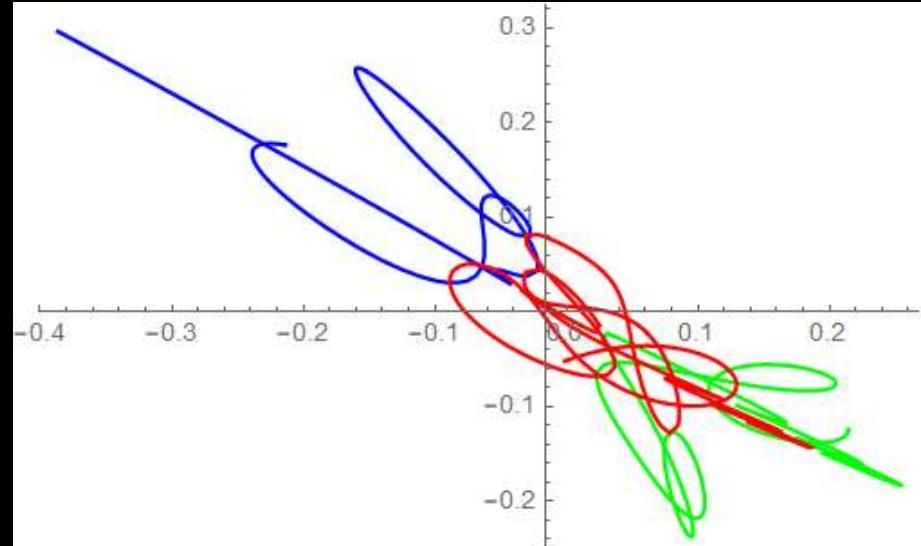
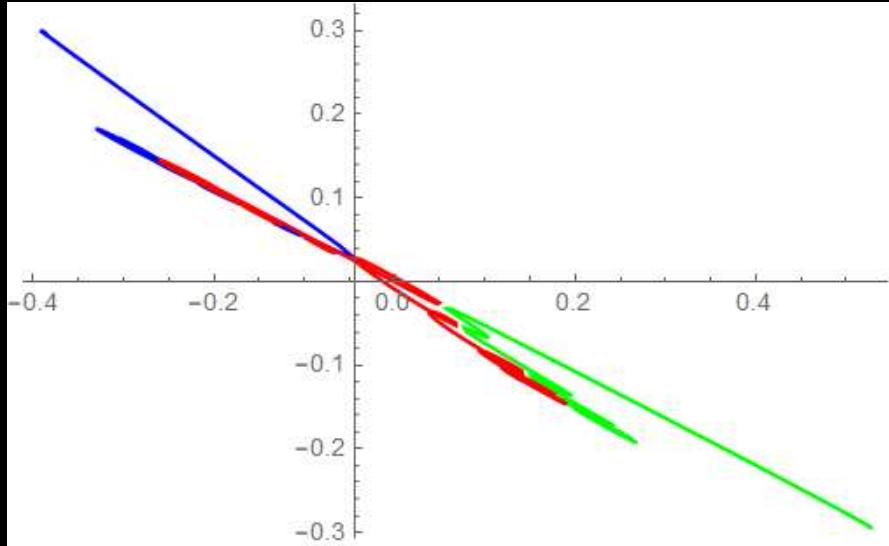
Mk^1_{Max}

Sh Sh_{Max} Mk^1 Mk_{Max} Mk_{Min}

1.09805, 1.09805, 1.0963, 1.0963, 0.7784

1,1,1,1,1,1,2,1,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,1,2,1,1,1,2,1,2,1,2,1,2,1,
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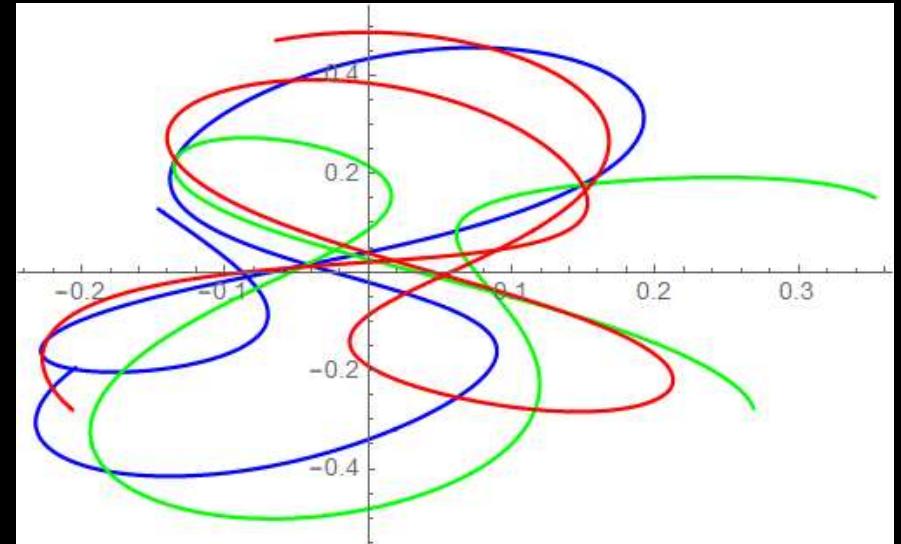
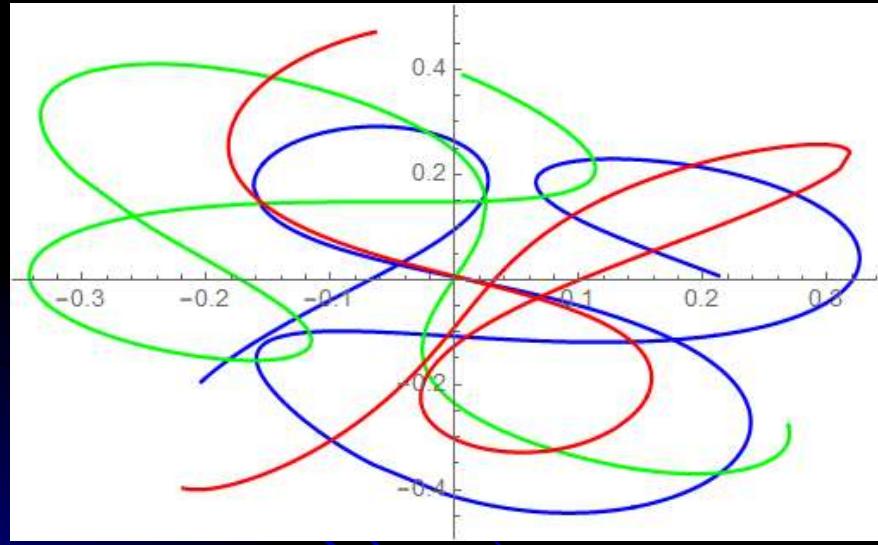
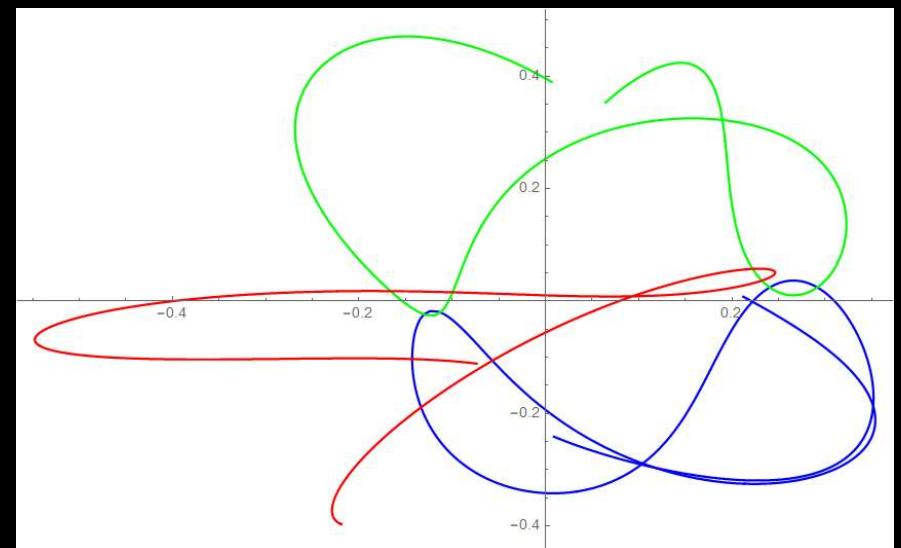
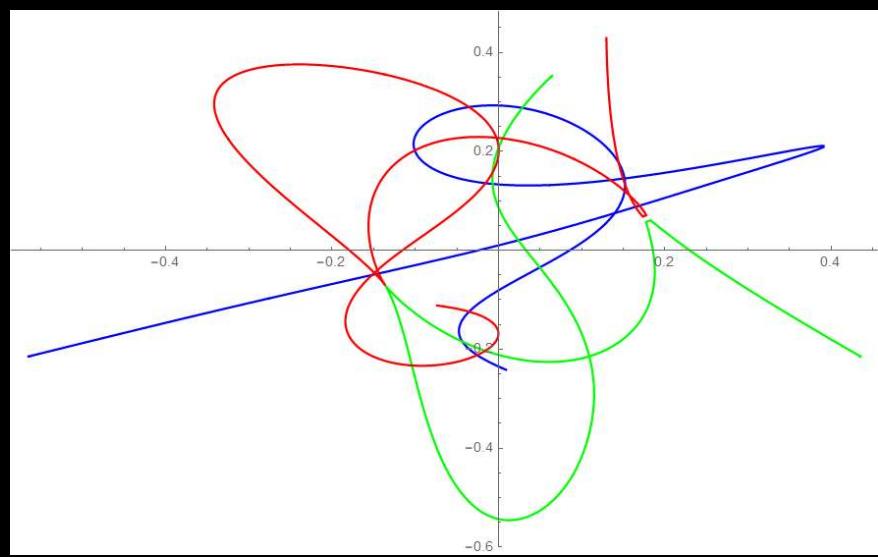
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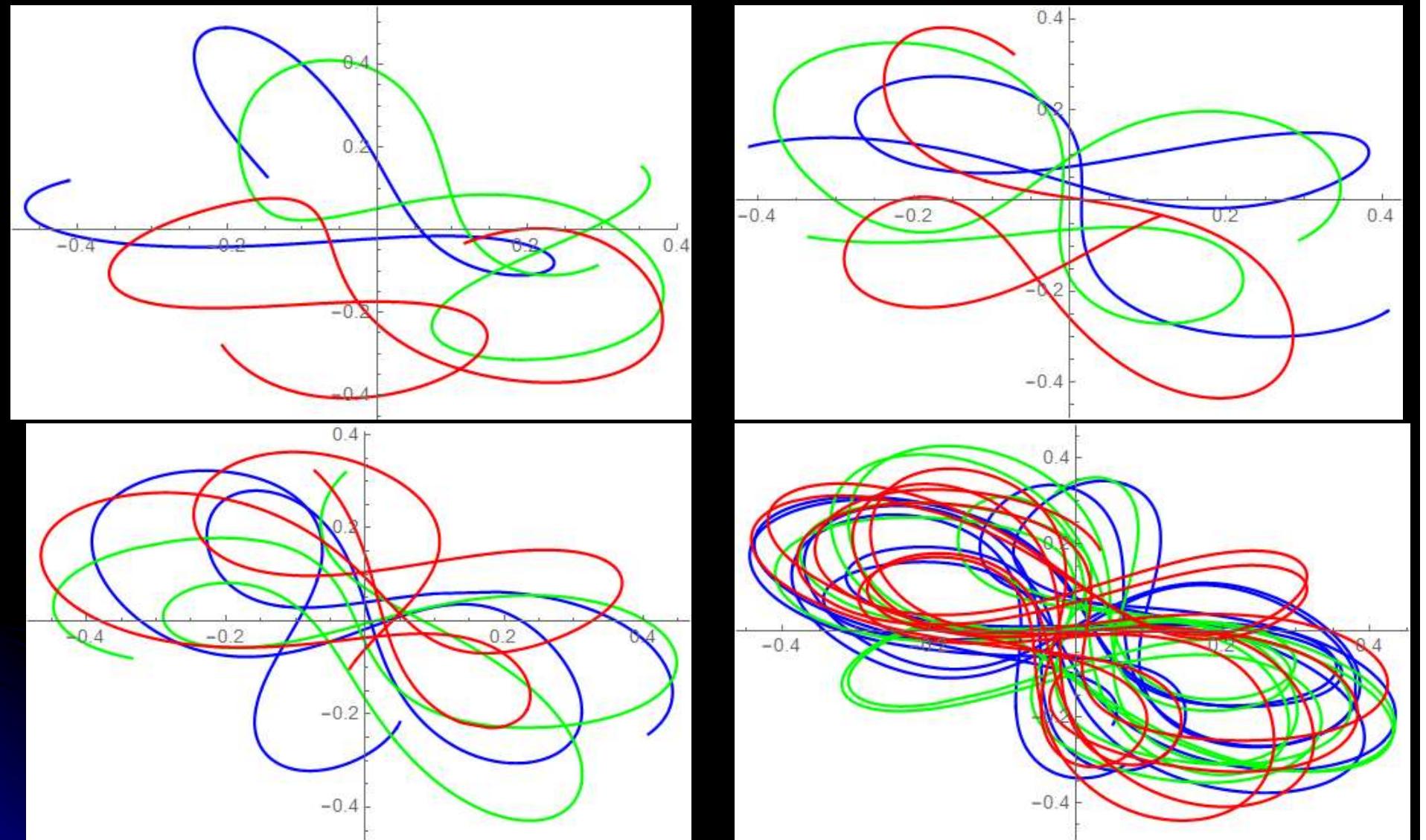
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Mk_{Max}Max

Sh	Sh _{Max}	Mk ¹	Mk _{Max}	Mk _{Min}
1.09855,	1.09861,	0.578,	1.0985,	0.578
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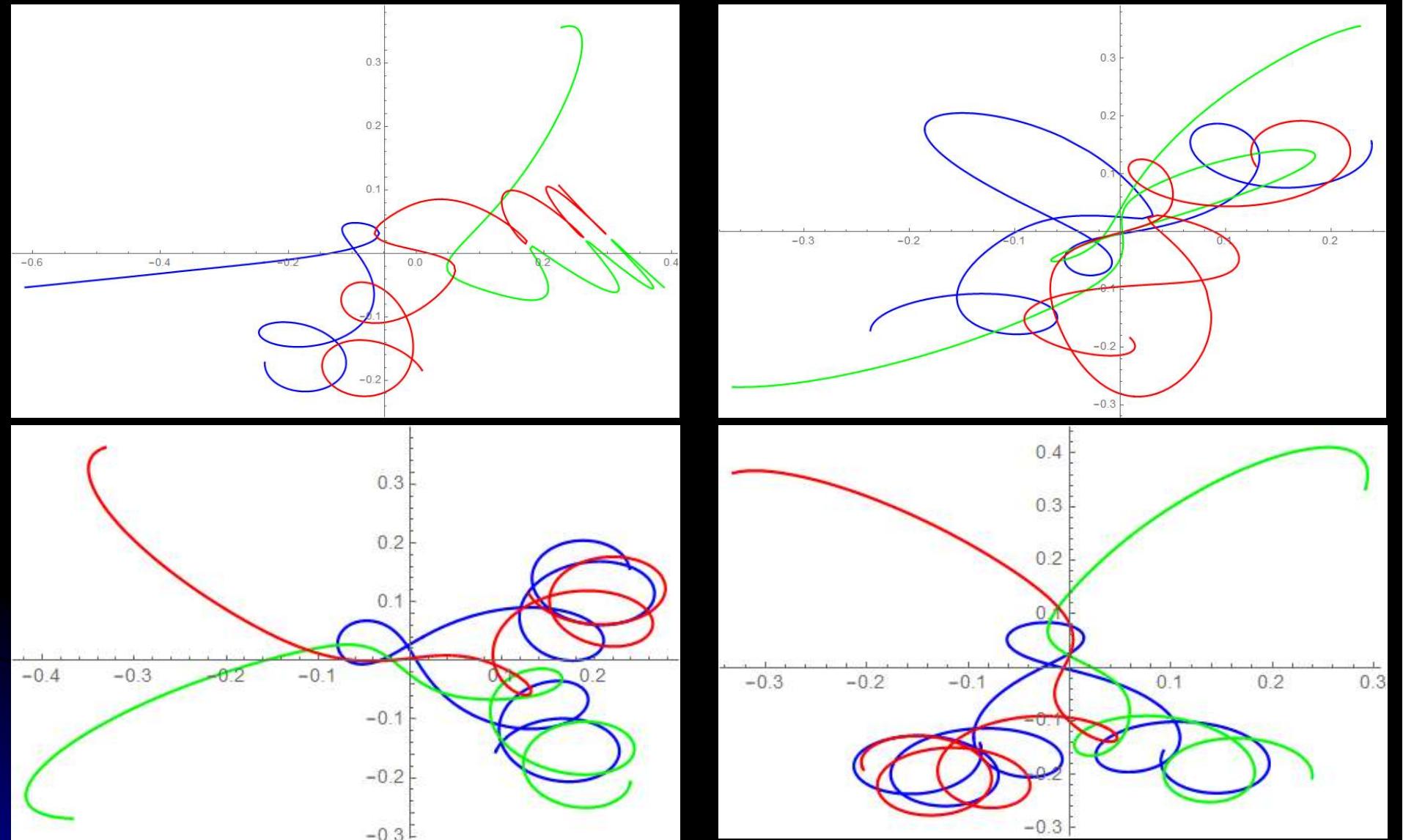
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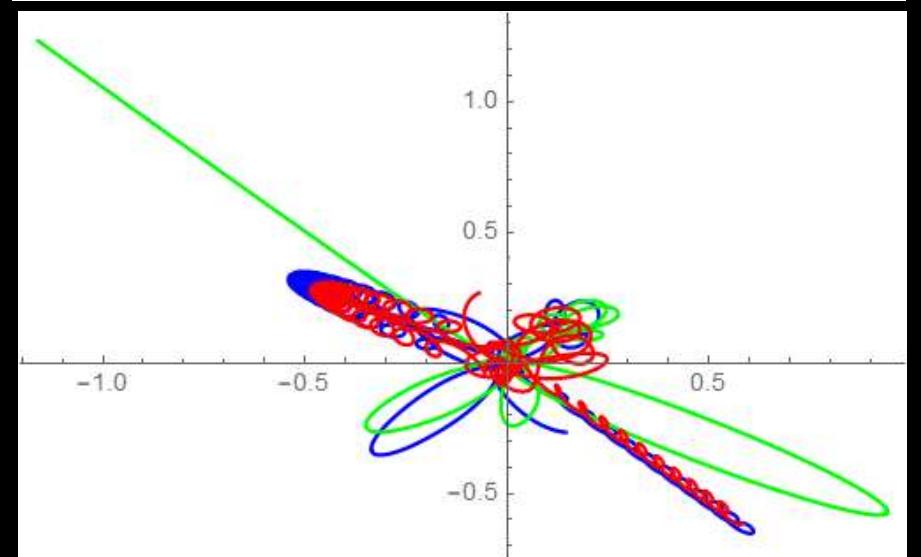
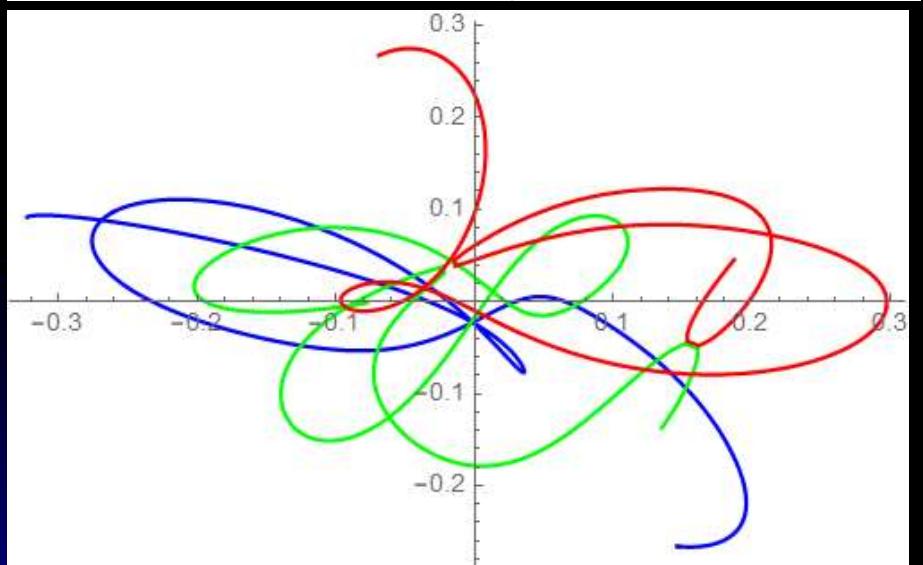
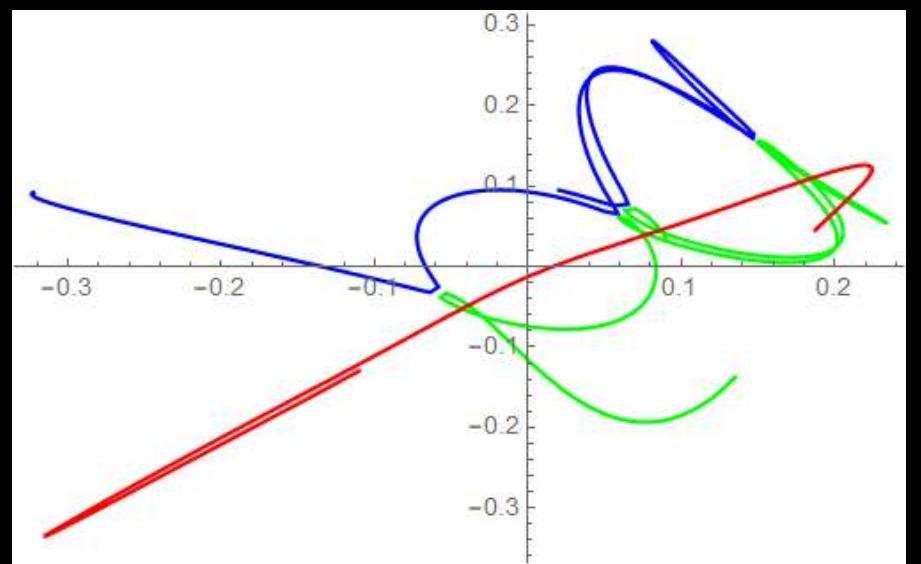
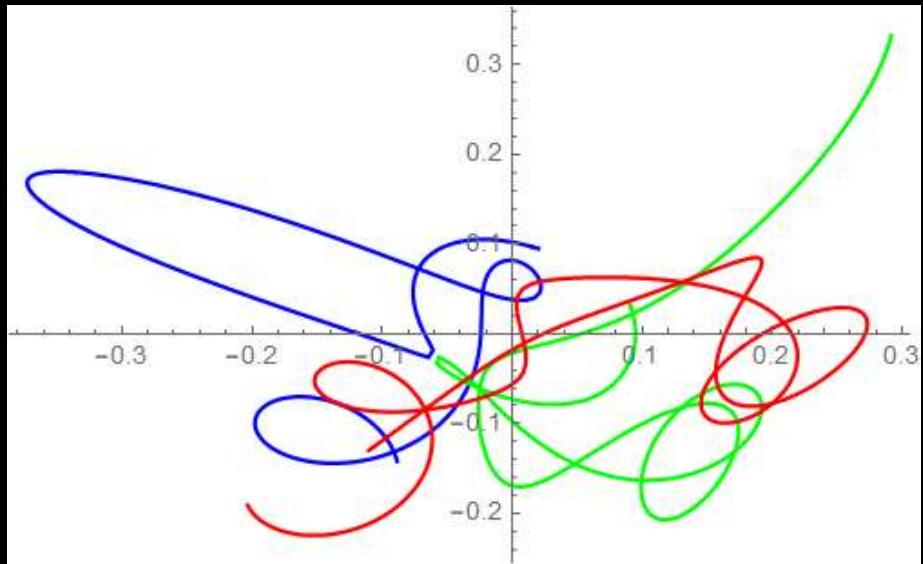
Sh	Sh_{Max}	Mk^1	Mk_{Max}	Mk_{Min}
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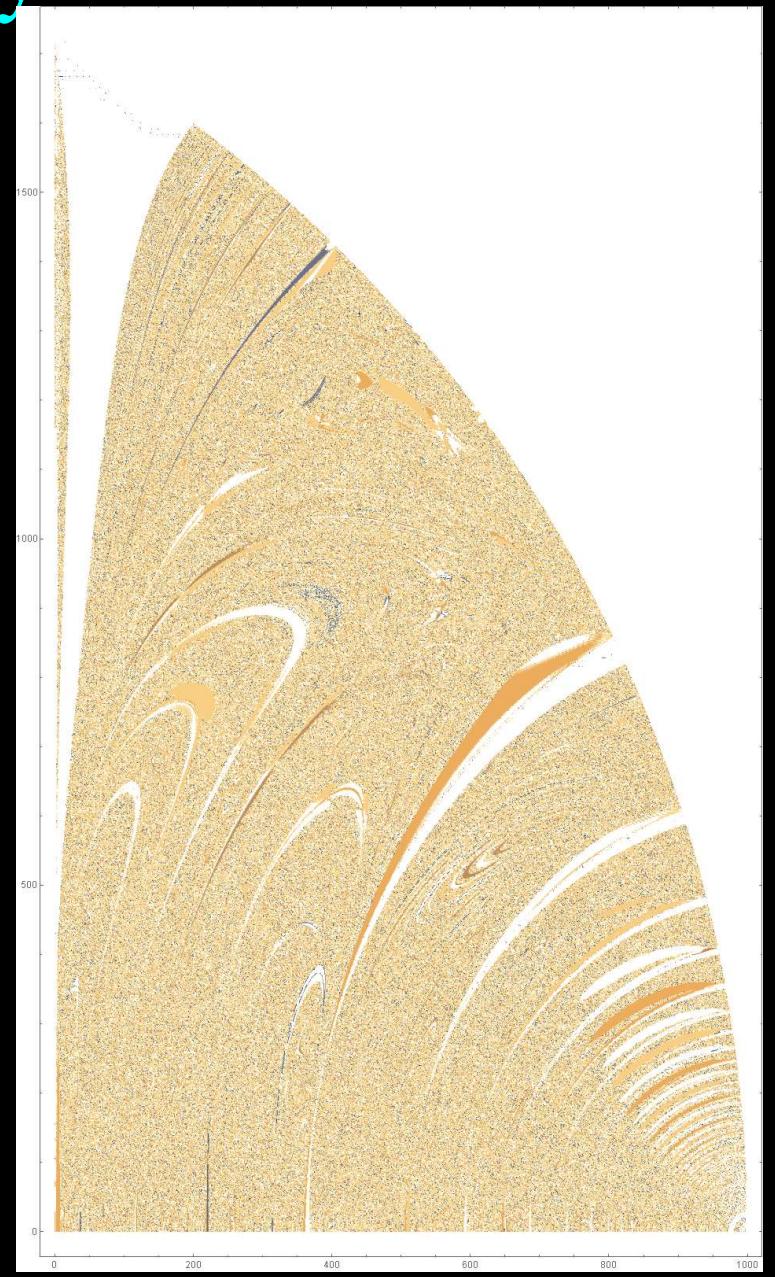
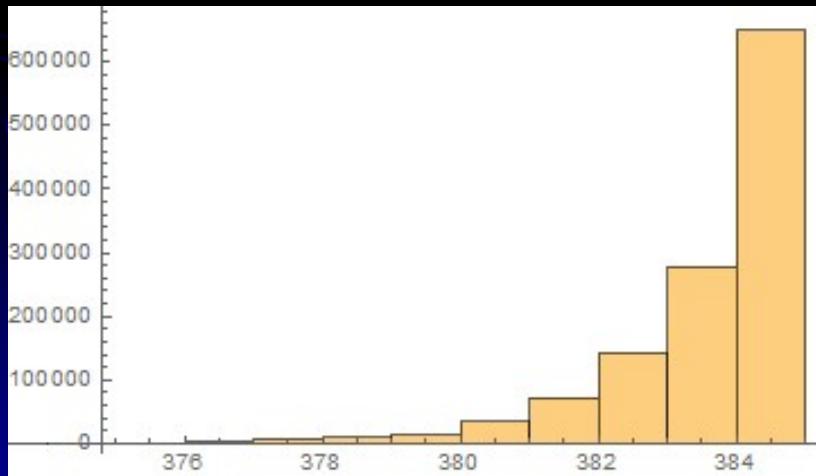


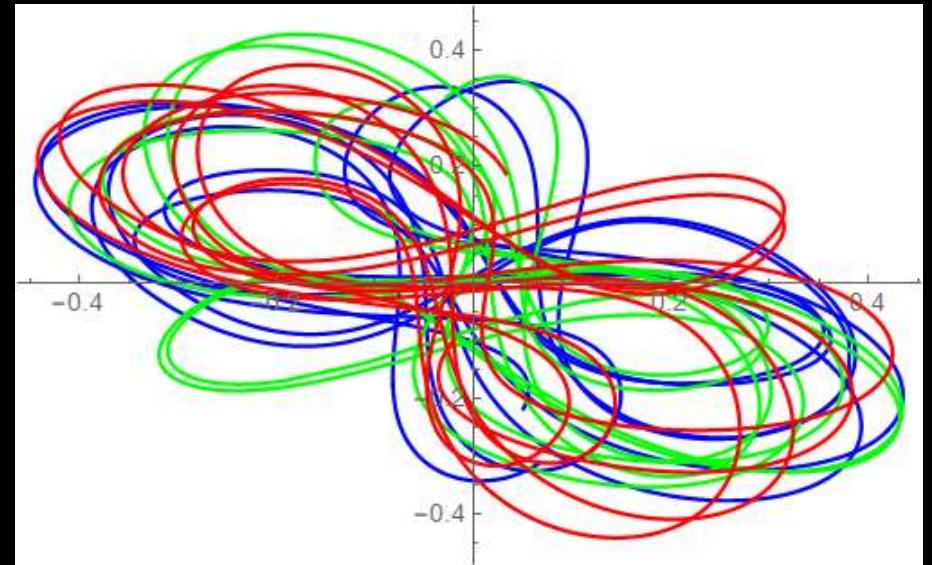
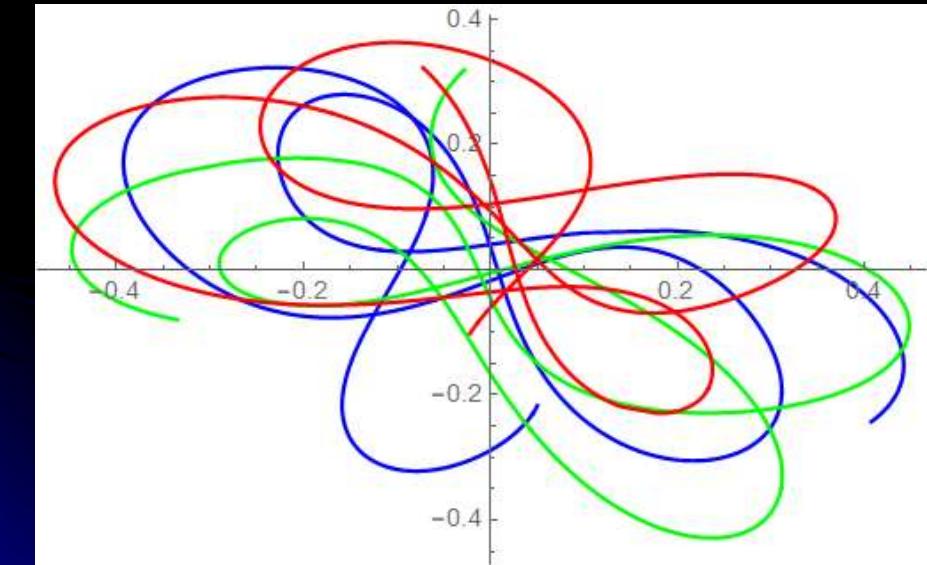
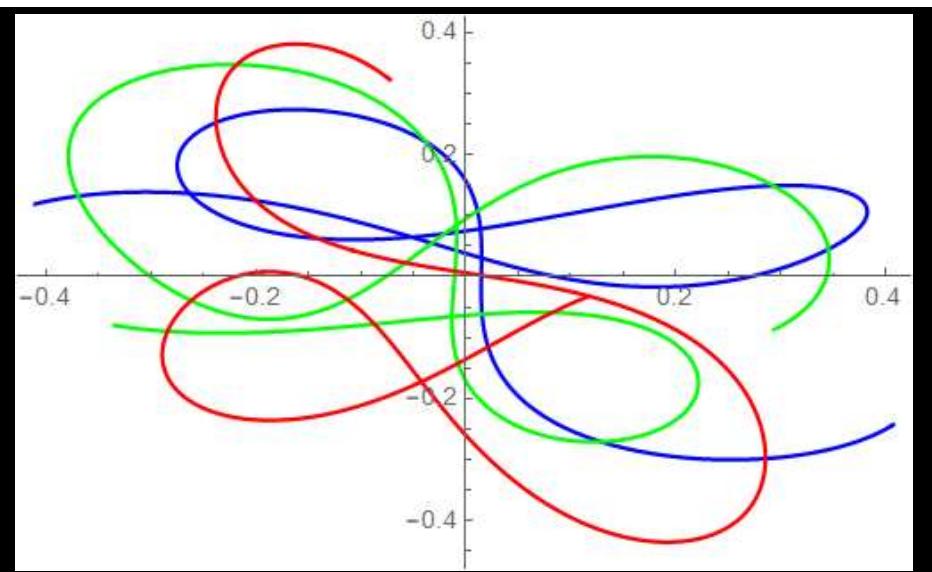
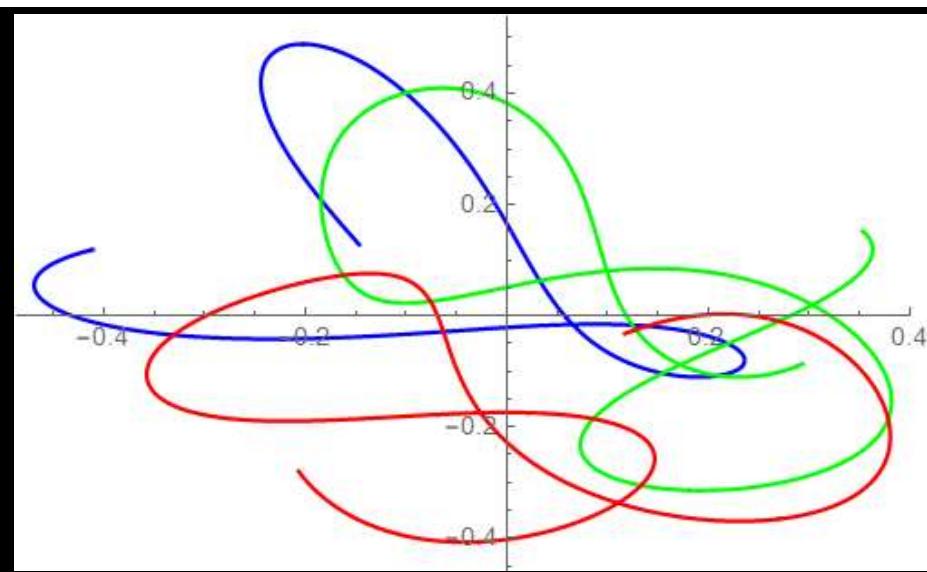
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What next?

Arnold complexity

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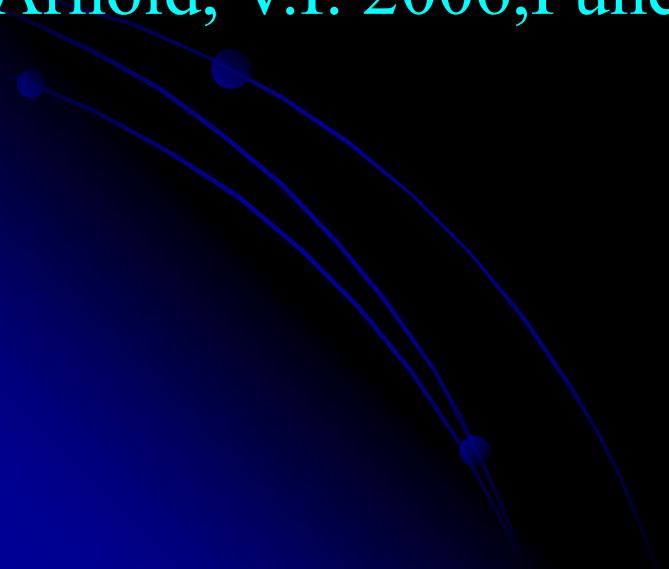
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NUOVO CIMENTO VOL. 108 B, N. 1, pp. 83 – 92, 1993

Arnold complexity:

Arnold, V.I. 2006, Funct Anal. Other Math., 1: 1-15



Thank you for attention!

