Polynomial coefficients as traces and applications to graph colorings

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Abstract. Let $G = C_n \times C_m$ be a toroidal grid (that is, 4-regular graph), where nm is even. We prove that this graph G is 3-choosable. We also prove some more general results about list colorings of direct products. The proofs are algebraic, the starting point is Alon–Tarsi application of Combinatorial Nullstellensatz, and the main difficulty is to prove that the corresponding coefficient of the graph polynomial is non-zero.

Let $\mathbb{F}$ be a field, $\mathbf{x} = (x_1, \ldots, x_n)$ a set of variables. For $A \subset \mathbb{F}$ and $a \in A$ denote

$$D(A, a) := \prod_{b \in A \setminus a} (a - b).$$

For a multi-index $\mathbf{d} = (d_1, \ldots, d_n) \in \mathbb{Z}^n_{\geqslant 0}$ denote $|\mathbf{d}| = d_1 + \ldots + d_n$, $\mathbf{x}^{\mathbf{d}} = \prod_{i=1}^n x_i^{d_i}$. For a polynomial $f \in \mathbb{F}[\mathbf{x}]$ denote by $[\mathbf{x}^{\mathbf{d}}]f$ the coefficient of monomial $\mathbf{x}^{\mathbf{d}}$ in polynomial f.

Choose arbitrary subsets $A_i \subset \mathbb{F}$, $|A_i| = d_i + 1$ for $i = 1, \ldots, n$. Denote $A = A_1 \times A_2 \times \ldots \times A_n$.

Recall the formula version of Combinatorial Nullstellensatz (it appeared in this form in quite recent papers [6, 8, 11], but essentially already in [5], see [7] for a modern exposition of the algebraic geometry behind this formula):

$$\left[\mathbf{x}^{\mathbf{d}}\right] f = \sum_{\mathbf{a} = (a_1, \ldots, a_n) \in A} \frac{f(\mathbf{a})}{\prod_{i=1}^n D(A_i, a_i)} \tag{1}$$

for any polynomial $f \in \mathbb{F}[\mathbf{x}]$ such that $\deg f \leqslant |\mathbf{d}|$.

In particular, if $[\mathbf{x}^{\mathbf{d}}]f \neq 0$, then (1) yields the existence of $\mathbf{a} \in A$ for which $f(\mathbf{a}) \neq 0$. This is Combinatorial Nullstellensatz [1], which has numerous applications.

Alon and Tarsi [2] suggested to use it for list graph colorings. Namely, if $G = (V, E)$ is a non-directed graph with the vertex set $V = \{v_1, \ldots, v_n\}$ and the edge set E, we define its graph polynomial in n variables $x_1, \ldots, x_n$ as

$$F_G(\mathbf{x}) = \prod_{(i,j)\in E} (x_j - x_i).$$

Here each edge corresponds to one linear factor $x_j - x_i$, so the whole F_G is defined up to a sign. Assume that each vertex v_i has a list A_i consisting of $d_i + 1$ colors, which are real numbers. A *proper list coloring of G subordinate to lists* $\{A_i\}_{1\leqslant i\leqslant n}$ is a choice of colors $\mathbf{a} = (a_1, \ldots, a_n) \in A_1 \times \ldots \times A_n = A$ for which neighbouring vertices have different colors: $a_i \neq a_j$ whenever $(i, j) \in E$. In other words, a proper list coloring is a choice of $\mathbf{a} \in A$ for which $F_G(\mathbf{a}) \neq 0$. If $|\mathbf{d}| = |E|$, the existence of a proper list coloring follows from $[\mathbf{x^d}]F_G \neq 0$.

Define the *chromatic number* $\chi(G)$ of the graph G as the minimal m such that there exists a proper list coloring of G subordinate to equal lists of size m: $A_i = \{1, \ldots, m\}$. Define the *list chromatic number* $\mathrm{ch}(G)$ of the graph G as the minimal m such that for arbitrary lists A_i, $|A_i| \geqslant m$, there exists a proper list coloring of G subordinate to these lists. Define the *Alon–Tarsi number* $AT(G)$ of the graph G as the minimal k for which there exists a monomial $\mathbf{x^d}$ such that $\max(d_1, \ldots, d_n) = k - 1$ and $[\mathbf{x^d}]F_G \neq 0$.

From above we see that the list chromatic number does not exceed the Alon–Tarsi number:

$$\mathrm{ch}(G) \leqslant \mathrm{AT}(G). \tag{2}$$

Further we consider the Alon–Tarsi numbers for the graphs which are direct products $G_1\square G_2$ of simpler graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$. Recall that the vertex set of $G_1\square G_2$ is $V_1 \times V_2$ and two pairs (v_1, v_2) and (u_1, u_2) are joined by an edge if and only if either $v_1 = u_1$ and $(v_2, u_2) \in E_2$ or $v_2 = u_2$ and $(v_1, u_1) \in E_1$.

It is well known (Lemma 2.6 in [10]) that $\chi(G_1\square G_2) = \max(\chi(G_1), \chi(G_2))$. Much less is known about the list chromatic number (and the Alon–Tarsi number) of the Cartesian product of graphs. Borowiecki, Jendrol, Král, and Miškuf [3] gave the following bound:

Theorem 1 ([3]). *For any two graphs G and H,*

$$\mathrm{ch}(G\square H) \leqslant \min(\mathrm{ch}(G) + \mathrm{col}(H), \mathrm{col}(G) + \mathrm{ch}(H)) - 1.$$

Here $\mathrm{col}(G)$ is the *coloring number* of G, i.e. the smallest integer k for which there exists an ordering of vertices $v_1, \ldots, v_n$ of G such that each vertex v_i is adjacent to at most $k - 1$ vertices among $v_1, \ldots, v_{i-1}$.

Our first result [9] concerns the toroidal grid $C_n\square C_m$ (here C_n is a simple cycle with n edges)

Theorem 2. $\mathrm{AT}(C_n\square C_{2k}) = 3$.

[9] the right hand side of (1) in the necessary case was treated as a trace of the $(2k)$-th power of a certain matrix which for some lucky choice of the sets A_i's appeared to be Hermitian that almost immediately yields the result. This last phenomenon looks bit mysterious for us. We do not know whether it works

for other interesting classes of graphs. The different way to work with these traces was proposed in [4]. It allowed to prove the following rather technical but general result.

Definition. We call a coefficient $\left[\mathbf{x}^{\xi}\right] F_G(\mathbf{x})$ of the graph polynomial F_G *central*, if $\xi_i = \deg_G(v_i)/2$ for all i, and *almost central*, if $|\xi_i - \deg_G(v_i)/2| \leqslant 1$ for all i.

Theorem 3. *Let G be a graph, all vertices in which have even degree. Suppose that the graph polynomial F_G has at least one non-zero almost central coefficient. Then for $H = G\square C_{2k}$ the central coefficient is non-zero. In particular, H is $(\deg_H/2+1)$-choosable and*

$$\operatorname{ch}(H) \leqslant \operatorname{AT}(H) \leqslant \frac{\Delta(H)}{2} + 1 = \frac{\Delta(G)}{2} + 2.$$

Note that Theorem 1 gives the bound $\operatorname{ch}(H) \leqslant \min(\operatorname{ch}(G) + 2, \operatorname{col}(G) + 1)$ under the same conditions. When $\operatorname{ch}(G)$ (or $\operatorname{col}(G)$) is small, this bound is stronger. But it can also be weaker when $\operatorname{ch}(G)$ and $\operatorname{col}(G)$ are close to $\Delta(G)$. For example, if $G = C_{2l+1}$ is an odd cycle, then F_G obviously has a non-zero almost central coefficient, so, by Theorem 3, $\operatorname{ch}(C_{2l+1}\square C_{2k}) \leqslant 3$ (so this reproves the main result of [9] by a different argument). On the other hand, Theorem 1 gives only $\operatorname{ch}(C_{2l+1}\square C_{2k}) \leqslant 4$.

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