On extensions of canonical symplectic structure from coadjoint orbit of complex general linear group.

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Symplectic structure of phase spaces of Isomonodromic Deformation Equations.

Originally the phase spaces of the Isomonodromic Deformation Equations are some algebraic symplectic spaces constructed from the orbits of the coadjoint group action.

The evolution of the theory generate a need for the extension of the main object that is the coadjoint orbit of the complex general linear group.

It is necessary for the development of the tau-function theory particularly.

Lie-Poisson-Kirillov-Kostant structure and Hamiltonian reduction of $\mathrm{T}^*\mathrm{GL}(N)$.

Consider $\mathrm{T}^*\mathrm{GL}(N) \ni (g, B)$, it is a symplectic space

$$\omega_{T^*} = \mathrm{tr}\, dB \wedge dg, \;\; B \in \mathrm{gl}^*(N), g \in \mathrm{GL}(N).$$

An action $(g, B) \to (ag, Ba^{-1})$ by the left shifts is the Poisson action, momentum map is $(g, B) \xrightarrow{\mu} gB \in \mathrm{gl}^*(N)$. Consider a subgroup $\mathrm{GL}_\Lambda(N) \subset \mathrm{GL}(N)$ of matrices commuting with $\Lambda \in \mathrm{gl}^*(N)$, and a constant value level $\cup_g(g, g^{-1}\Lambda)$ of the momentum map.

This subgroup acts on the momentum level by the left shifts. A quotient with respect to this action

$$(g, g^{-1}\Lambda) \sim (g_\Lambda g, g^{-1}\Lambda g_\Lambda^{-1}) = (g_\Lambda g, (g_\Lambda g)^{-1}\Lambda), g_\Lambda \in \mathrm{GL}_\Lambda(N)$$

, is a symplectic space with the Lie-Poisson-Kirillov-Kostant form:

$$\omega_{LP} = \mathrm{tr}(dg^{-1}\Lambda) \wedge dg.$$

Bertola&Korotkin extension.

The quotient space $(g, g^{-1}\Lambda) \sim (g_\Lambda g, (g_\Lambda g)^{-1}\Lambda)$ is isomorphic to the coadjoint orbit $\mathcal{O} := \cup_g g^{-1}\Lambda g$.
How to extend it?
Trivial: $\mathrm{T^*GL}(N) \supset (g, g^{-1}\Lambda)$.
Non-trivial extension by M.Bertola and D.Korotkin:

$$\widetilde{\mathcal{O}}_{BK} := \bigcup_{g\in \mathrm{GL}(N), \Lambda\in\Delta} (g, g^{-1}\Lambda), \quad \omega_{BK} = \mathrm{tr}\, d(g^{-1}\Lambda) \wedge dg.$$

It is symplectic for Λ with different eigenvalues only, because this 2-form ω_{BK} is degenerated if eigenvalues Λ coincide.

I introduce a non-trivial extension of $\mathcal{O}$ for any Jordan structure of Λ in my talk.

Flag structure defined by $\mathcal{O}$.

Main object is a flag structure defined by $A \in \mathcal{O}$.
In case of different eigenvalues the flag in question is a series of sums of eigenspaces $F_k = \sum_{i=1}^{k} \ker(A - \lambda_i \mathrm{I})$.
In general case the same object can be defined

$$F_k := \ker \prod_{i=1}^{k} (A - \lambda_i \mathrm{I})$$

The central role in the extension itself play (abstract) linear spaces $F_k/F_{k-1} =: [F_k]$ and spaces of their linear transformations

$$\mathrm{End}\ [F_k].$$

In the case of general position these spaces are isomorphic to the eigenspaces. Denote their dimensions by

$$n_k := \dim[F_k].$$

Orbit as fiber-bundle.

In order to connect $\mathrm{End}\,[F_k]$ with the orbit we need to construct a fiber-bundle

$$\mathcal{O} \to \mathrm{Gr}^{\Phi}_{\{\vec{n_\circ}\}},$$

where $\mathrm{Gr}^{\Phi}_{\{\vec{n_\circ}\}} \ni 0 \subset F_1 \subset \cdots \subset F_{M-1} \subset F_M = V \simeq \mathbb{C}^N$ is a flag Grassmanian.
Let us define a complementary flag

$$V_k \subset V_{k-1}, V_k := V/F_k, \;\; V_0 = V, V_M = 0.$$

A fiber of the fiberbundle is isomorphic to an open set of

$$\mathrm{Hom}(V_k, [F_k])$$

Let us introduce coordinates. We need to cover the base by overlapping open subsets and define transition functions.

Trivialization of the bundle $\mathcal{O} \to \mathrm{Gr}^{\Phi}_{\{\vec{n_\circ}\}}$.

Let $\{\mathbf{e}_i\}_{i=1}^N$ be a standard basis of $\mathbb{C}^N$. Let L_k be a linear combination of n_k subsequent basic vectors

$$L_k := \mathscr{L}(\mathbf{e}_{S_k+1}, \dots, \mathbf{e}_{S_k+n_k}),\ S_k = n_1 + n_2 + \cdots + n_{k-1},$$

$$L_k \simeq [F_k],\ E_k := \oplus_{i=k+1}^{M} L_i \simeq V_k.$$

A coordinate domain of the flag coordinates form such $A \in \mathcal{O}$ that for all subspaces F_k

$$F_k \cap E_k = 0,\ \text{or equivalently}\ V = F_k \oplus E_k.$$

An atlas that cover the whole orbit consists of the charts of the same construction but with the permuted basic vectors $\mathbf{e}_i$. The charts are parameterized by the elements of the symmetric group $\Sigma_N \ni \alpha$.

Coordinate mappings P_k, Q_k.

Let $\mathcal{A} \in \mathrm{End}\ V$ be a linear transformation that has matrix A in the basis $\{\mathbf{e}_i\}_{i=1}^N$. Transformation $\mathcal{A}$ can be contracted on the factor-spaces V_k. We denote the contractions by $\mathcal{A}_k \in \mathrm{End}\ V_k$. The factor-space $[F_k]$ can be considered as n_k-dimensional submanifold of V_{k-1}, it is the eigenspace of $\mathcal{A}_{k-1}$ corresponding to λ_k.

Consider a coordinate chart that identify $E_k \simeq V_k$. The position of $[F_k]$ is determined by $Q_k \in \mathrm{Hom}([F_k], V_k)$. It is the projections of the points of $[F_k]$ to L_k parallel to E_k:

$$Q_k(x) + x \in L_k, \quad x \in [F_k], \quad Q_k(x) \in E_k,$$

it is a coordinate on the base.

To get coordinates on the fibre we contract $\mathcal{A}_{k-1} - \lambda_k \mathrm{I}$ on V_k and set $P_k \in \mathrm{Hom}(V_k, [F_k])$ as a projection of its action to $[F_k]$ parallel to E_k.

Transition functions.

We need transition functions to complete the construction of the orbit $\mathcal{O}$ as the fiber bundle. Different charts of the atlas differ by the directions of the projection of the same $(\mathcal{A}_{k-1} - \lambda_k \mathrm{I})|_{V_k}$ on the same $[F_k]$. The difference in question is the same for all $\mathcal{A}$ with the same projection $\mathcal{A}_k$ due to Thales theorem.

Let us denote the directions of the projections for the charts α and β by E_k^α and E_k^β. Let us denote

$$\Phi_k^{\alpha\beta}(A_k) := P_k^\beta(A) - P_k^\alpha(A) \in \mathrm{Hom}(V_k, [F_k]).$$

We treat this transformation as a transition function. It transforms the fiber in the following way:

$$P_k^\beta(A) = P_k^\alpha(A) + \Phi_k^{\alpha\beta}(A_k).$$

Coordinates on $[F_k]$.

Every linear space $[F_k]$ has basis defined by the columns of g, $A = g^{-1}\Lambda g$. These columns are the representatives of the elements of the quotient-spaces $[F_k]$.
The coordinates on the quotient spaces are constructed at all charts $\alpha \in \Sigma_N$. The coordinates are defined by projection of the basis from the coordinate subspace L_k^α to $[F_k]$ parallel to E_k^α.
The transition function $\phi_{GL}^{\alpha\beta} \in GL([F_k])$ is defined as a linear transformation of the basis constructed using E_k^α, L_k^α to the basis constructed by E_k^β, L_k^β on $[F_k]$. The bases are the projections of the bases from L^α and from L^β parallel to E_k^α and to E_k^β correspondingly.

Extension of orbit.

An extension of the orbit has two conjugated components. First one is an arbitrary linear transformation

$$B_k \in \mathrm{End}\,[F_k],$$

and the second one is a point of its cotangent bundle

$$R_k \in \mathrm{T}^*_{B_k}\mathrm{End}\,[F_k].$$

The coordinates of the same pair $(B_n, R_n) \in \mathrm{T}^*\mathrm{End}\,[F_k]$ in different charts $\alpha, \beta \in \Sigma_N$ are connected by the similarity transformations $\phi^{\alpha\beta}_{GL}$.

$$\omega^{ext} = \omega_{LP} + \omega^{add}, \quad \omega^{add} = \sum_{k=1}^{M} \omega_k, \quad \omega_k = \mathrm{tr}\, dR_k \wedge dB_k$$

The End.

Thank You!:)