Growth in groups and the number of curves and knots

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Abstract. We use results of Vershik, Nechaev, and Bikbov on growth of random heaps to improve known lower bounds on the rate of growth of the number of knots with respect to the crossing number.

We study the structure and statistical characteristics of the set of classical knots (see [M15, M18, M18b, M19, BM19] and references therein). A particular point of this study is the growth rate of the number of knots with respect to various complexity measures on the set of knots. Historically, the crossing number is considered as the most natural knot complexity measure. The growth rate of the number of knots with respect to the crossing number is studied in particular in [ES87, W92, STh98, Th98, St04, Ch18]. Being applied to the sequence $(K_1, K_2, K_3, ...)$, where K_n denote the number of knots of n crossings,

results of [ES87] imply that[1]

$$2.13... \quad \leq \quad \liminf_{n\to\infty} \sqrt[n]{K_n};$$

results of [W92] imply that

$$\limsup_{n\to\infty} \sqrt[n]{K_n} \quad \leq \quad 13.5;$$

results of [STh98] imply that

$$\limsup_{n\to\infty} \sqrt[n]{K_n} \quad \leq \quad \frac{101+\sqrt{21001}}{20} = 12.29...;$$

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[1]The lower bound $2.68 \leq \liminf_{n\to\infty} \sqrt[n]{K_n}$ given in [W92] as an interpretation of results obtained in [ES87] seems to be a typo.

and results of [St04] imply that

$$\limsup_{n\to\infty} \sqrt[n]{K_n} \quad \leq \quad \frac{91+\sqrt{13681}}{20} = 10.39...$$

Thus, the record asymptotic estimates, presented in the literature, for the growth rate of K_n are

$$2.13... \quad \leq \quad \liminf_{n\to\infty} \sqrt[n]{K_n} \quad \leq \quad \limsup_{n\to\infty} \sqrt[n]{K_n} \quad \leq \quad 10.39...$$

For some reasons explained in [M18] it would be useful to find bounds a and b such that

$$a \quad \leq \quad \liminf_{n\to\infty} \sqrt[n]{K_n} \quad \leq \quad \limsup_{n\to\infty} \sqrt[n]{K_n} \quad \leq \quad b$$

and

$$a^3 \quad > \quad b^2.$$

It turns out that a new lower bound

$$4 \quad \leq \quad \liminf_{n\to\infty} \sqrt[n]{K_n}$$

is implied by the results of [V00, VNB00] on the growth rate of locally free semigroups (heaps). To obtain this bound, we construct embeddings of locally free semigroups into the set of knots. Furthermore, passing to more complex semigroups with weighted elements, we show that

$$4.45 \quad \leq \quad \liminf_{n\to\infty} \sqrt[n]{K_n}.$$

Moreover, we have the same lower bound for the case of alternating prime knots.

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