Dynamical systems with a quadratic right-hand side

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Abstract. Difference schemes for dynamical systems with a quadratic right-hand side, setting a one-to-one correspondence between initial and final values (reversible difference schemes), are considered in the report. The results of computer experiments with these schemes are presented.

Most systems of ordinary differential equations arising in applications cannot be integrated in a finite form. The idea of finding and classifying all differential equations that are integrable in a finite form arose in the 19th century, but the very concept of integrability in a finite form admits a lot of interpretations: from integration in elementary functions to finding differential equations with the Painlevé property.

Among these interpretations, in our opinion, of particular interest is the purely algebraic approach proposed in the early works of Painlevé [1, 2]. In these works, it is proposed to find and integrate all differential equations, the general solution of which defines a birational or at least algebraic correspondence between the initial and final values. All linear differential equations define a linear correspondence between initial and final values. Painlevé sought to prove that among nonlinear ones, birational correspondences are given only by those that integrate in Abelian functions and their degenerations or are reduced to linear ones.

This approach seemed to us interesting because it does not require fixing the list of admissible transcendental functions [3]. The above algebraic property of the general solution itself singles out a certain class of transcendental functions, which surprisingly coincides with the set used in classical mechanics.

Since the approach is purely algebraic, it can be easily applied to finite differences. Let us first turn to the first-order differential equation

$$\frac{dx}{dt} = f(x,t)$$

with a rational right-hand side. On the straight line, any birational correspondence is a Möbius transform, and therefore the Riccati equation

$$\frac{dx}{dt} = p(t)x^2 + q(t)x + r(t) \tag{1}$$

and only the Riccati equation defines the birational correspondence between the initial and final values x.

According to the finite difference method, the differential equation is replaced by an algebraic equation connecting the value of the solution of this equation at time t with the value at $t + \Delta t$. Let us call the first value the initial value and denote it as x, and the second one call the final value and denote it as $\hat{x}$. Then the difference analogue of the Painlevé problem is as follows: find all differential equations that can be approximated by algebraic equations defining birational correspondences between x and $\hat{x}$. Since the group of birational transformations of the line does not change in any way under the transition to finite differences, the Riccati equation and only the Riccati equation admits such an approximation, namely

$$\hat{x} - x = (px\hat{x} + qx + r)\Delta t.$$

Thus, the discrete and continuous cases turn out to be identical [4].

However, when passing to systems of equations, the situation changes radically. Any system of m differential equations with a quadratic right-hand side is approximated by a difference scheme defining a birational correspondence between the initial and final values as points of m-dimensional projective spaces. To do this, it is necessary to replace the derivative with a finite difference, and the monomials $x_i x_j$ on the right-hand side with $\hat{x}_i x_j$.

On the other hand, autonomous systems of differential equations with a quadratic right-hand side are a very large class of nonlinear differential equations, to which all equations describing the motion of the top belong. Back in the 19th century, four cases were distinguished among them when the motion of the top is integrated in Abelian or elliptic functions. Therefore, the class of differential equations admitting an approximation defining a birational correspondence between initial and final values is much wider than the class of differential equations that themselves define a birational correspondence between initial and final values.

Here it is necessary to make one more clarification: for Abelian functions to appear in the theory, Painlevé considered differential equations defining birational correspondences between initial and final values as points on integral manifolds distinguished by algebraic integrals. In the finite difference case, we obtain birational correspondences between the initial and final values as points of the projective space without regard for the algebraic integrals.

Let us explain what has been said with the simplest example. Dynamic system

$$\dot{x} = y, \quad \dot{y} = 6x^2 - a \tag{2}$$

has an algebraic integral

$$\frac{y^2}{2} - 4x^3 + ax = C_1, \tag{3}$$

the meaning of which is the total mechanical energy. If the value of C_1 is fixed, then the general solution

$$x = \wp(t + C_2, 2a, C_1), \quad y = \wp'(t + C_2, 2a, C_1), \tag{4}$$

defines a birational correspondence between the initial point (x_0, y_0) and the final point (x_1, y_1) on the elliptic curve (3). This correspondence does not extend to a birational transformation of the projective plane xy. Nevertheless, we can approximate the system (2) by the difference scheme

$$\hat{x} - x = \frac{\hat{y} + y}{2}\Delta t, \quad \hat{y} - y = (6x\hat{x} - a)\Delta t \tag{5}$$

which defines a birational correspondence between the points (x, y) and $(\hat{x}, \hat{y})$ of the projective plane. In this case, the integral of motion (3) is not preserved. Thus, in the transition from a continuous model to a discrete one, the algebraic integral of motion is not preserved and thanks to this fact we move from the group of birational transformations of an elliptic curve to the group of Cremona transformations.

It should be noted that the group of birational transformations of the plane is very large, and its study was begun in the works of Cremona in the 1860s and is not finished now even in rough form. Hermite [5] believed that the theory of birational transformations of curves would be a section in the theory of the Cremona group, which F. Klein considered an annoying mistake, which he considered necessary to describe in detail in his Lectures on the History of Mathematics [6, ch. 7]. Now it turns out that Hermite was right after all, and there is a connection between birational transformations on curves and Cremona transformations, which manifests itself in the discretization of dynamical systems.

In the study of mechanical autonomous models, it is often noted that there must be a one-to-one correspondence between the initial and final data (the property of model reversibility). Birational correspondence is a special case of one-to-one correspondence. When passing to difference schemes, any mapping becomes algebraic; therefore, reversibility is equivalent to the property under discussion. In this case, it is easy to correct the scheme so that it has t-symmetry.

Since many of the differential equations with a quadratic right-hand side describe nonlinear oscillators, we decided to investigate the periodicity of approximate solutions in the Sage computer algebra system. For a linear oscillator, this issue was investigated earlier in [7].

First of all, we fixed the initial value and the natural number n and chose such a step Δt that in n steps the solution of system (2) according to the scheme (5) would exactly return to initial data. It turned out that this is possible for almost all n, and one n corresponds to several positive values of the step. The resources do not allow taking n large, but it was possible to observe the convergence of the product of n and the smallest positive step to the exact period.

Then we fixed the step Δt and the square on the phase plane xy. We filled this square evenly with points, which we used as initial ones for constructing solutions according to the reversible scheme (5).

In this phase portrait, special lines appeared, which we propose to call equiperiodic. Namely, the set of initial points that, for a given step Δt, give an approximate solution with a period n, will be called an equiperiodic set of the n-th order. It is easy to prove that equiperiodic sets are integral for approximate solutions. In the case of the scheme (5), such a quasiperiodic set is the curve $F_n(x, y,\ dt) = 0$. For small orders, we managed to calculate these curves, they turned out to be elliptical. Equiperiodic curves for the exact solution of the system (2) are elliptic curves of the bundle (3). When discretized, this linear family turns into a countable set of equiperiodic curves $F_1, F_2, \dots$. Thus, the integral of motion generates a bundle of integral curves, which becomes discrete upon discretization.

In conclusion, let us formulate the main results. First, the class of differential equations approximated by reversible separation schemes is much wider than the class of differential equations that specify birational correspondences between initial and final values. Second, the place of birational transformations on algebraic integral manifolds in the proposed theory of difference schemes is occupied by Cremona transformations. Therefore, the previously noted obstacles to the constitution of conservative explicit difference schemes [8] by no meas impoverish the algebraic properties of difference schemes, but only indicate significant differences in continuous and discrete theories. The example considered shows that the question should be raised not about the inheritance of the algebraic properties of an exact solution by approximate ones, but about their transformation.

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