Mechanics, Physics and Arithmetic

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Abstract. The author's results on the exact solvability of the Euler-Poisson equations (from classical mechanics) in special L-functions (from modular arithmetic) are presented. The results are unexpected within the framework of the classical approach. This requires extensive reconcilement with the already accumulated results in this field (primarily with classical solutions and their chaotic perturbations), carried out in the monograph [1] (see also [2], [3]). The emerging general solution geometrically represents the potential of a canonical multivalued finitely generated analytic map of centrally-similar rotation in the Euclidean 3d-space with a natural Galois group factor-structure.

The fundamental mechanical meaning of "analytical arithmetic solutions" is the special self-oscillatory modes of the generalized (super) gyroscopic dynamics of heavy tops (special relativistic quantum oscillations), obtained on the base of the Kovalevskaya method for solving the original equations and is lost by classical mechanics due to its nonrelativism (which includes classical solidity of the configuration space, but excludes canonical solidity of the phase space, induced by the involution of time reversibility of the original equations).

The obtained theoretical results agree with the experimental data of the fundamental Dzhanibekov effect, clearly presented in the computer visualization of paper [4], which also demonstrates the visualization of the Galois axis, a new object for classical rigid body dynamics.

Euler-Poisson equations and their exact solution

The Euler-Poisson equations describing the dynamics of three-dimensional heavy solids (tops) in a plane-parallel gravitational field have the form:

$$d\overrightarrow{M}/dt = \left[\overrightarrow{M}, \overrightarrow{\omega}\right] + k\left[\overrightarrow{\gamma}, \overrightarrow{c}\right], \tag{1}$$

$$d\overrightarrow{\gamma}/dt = \left[\overrightarrow{\gamma}, \overrightarrow{\omega}\right], \tag{2}$$

- where the corresponding vectors $\overrightarrow{M} = I\overrightarrow{\omega}$ are the kinetic moment of the body, $\overrightarrow{\omega}$ is the angular velocity of the body, $\overrightarrow{c}$ is the directing vector of the line from the

fixed point to the center of mass, $\overrightarrow{\gamma}$ is the projections of the vertical unit vector on the axis of the moving coordinate system, rigidly connected with the body;
- where also I is the diagonalized inertia tensor of the body at its fixed point, $k{=}m\boldsymbol{g}|r_c|$ is the coefficient equal to the product of the body weight by the distance from the fixed point to the center of mass, [,] is the operation of the vector product in three-dimensional Euclidean space.

Equations (1)-(2) locally (over $\mathbb{R}$) are analytic differential equations in the classical sense: their coefficients and variables are real-analytic functions of real time $t{=}\mathbb{R}$.

It turns out (see [1]) that the classical equations (1)-(2) define a new class of special functions are defined by their general solution representing the full space of invariant (central) functions on the canonical simply connected functional globally analytic extension of the group $SO(3{,}\mathbb{R})$:

1. representing the "affine differential integral" on "the standard" functional orthogonal group $SO_{an}(3, \mathbb{C})$ - $\mathbb{C}$ -analytic 3d-ball;
2. generated by the canonical group map of the analytic adjoint group rolling of the standard 3d-sphere in the space $\mathbb{E}^3(\mathbb{C})$ and denoted by $SO_{an}(3{,}\mathbb{C})$.

Theorem 1. Analytical differential equations (1)-(2) have an additional integral (invariant), functionally independent with classical integrals (invariants), of the form

$$F = \exp\left(t^2 - {\omega_3}^2 {-} {\omega_2}^2 - {\omega_1}^2 - {\gamma_3}^2 {-} {\gamma_2}^2 - {\gamma_1}^2\right)$$

and representing the canonical potential of a canonical group continuous centrally-similar rotation in the space $\mathbb{E}^3(\mathbb{C})$.

An additional invariant F allows us to precisely solve equations (1)-(2), since it has the meaning of a canonical global metric on a group $SO(3)$.

Theorem 2. The general solution of the differential equations (1)-(2) describing the dynamics of the kinetic moment vector of heavy tops $\overrightarrow{M}(s/q/t)$, where $t \in \mathbb{R}$, $q \in \mathbb{H}^+$, $s \in \mathbb{C}$, is represented by the canonical coordinates on analytical group $SO_{an}(3, \mathbb{C})$ in the following forms:

$$\overrightarrow{M}(s/q)_{general} = \exp\left(\zeta\left(s{,}\Delta_{12}\left(q\right)\right)\right) \ , \tag{3}$$

$$\overrightarrow{M}(\mathrm{s}/q/t)_{general} = \exp\left(\zeta\left(\mathrm{s}{,}\Delta_{12}\left(q\right) \otimes_{\mathrm{SL}_2(\mathbb{Z})} PGL_2\left(\mathbb{Q}\left(s/t\right)\right)\right)\right), \tag{4}$$

$$\overrightarrow{M}(t)_{general} = \exp\left(\zeta\left(\tfrac{1}{2}{+}it, \Delta_{12}\left(q\right)\right)\right) \tag{5}$$

The function $\zeta\left(s{,}\Delta_{12}\left(q\right)\right)$ represents

- the canonical global (1-map) metric on 3d-sphere $\mathbb{S}^3$;

- the canonical metric in the phase space of equations (1)-(2);

- the potential of multivalued continuous monodromic dynamics of tetrahedra accompanying analytic tops;

- the Hamiltonian of canonical (vertical) equilibrium Hamiltonian of globally (over $\mathbb{C} \cup \infty$) analytical mathematical pendulum;

- the universal potential of the modular parametrization of elliptic curves over $\mathbb{Q}$;

- the canonical periodic function with three pure imaginary periods - canonical periods of 1) the "universal accompanying trihedron dynamics for analytic tops", 2) the continuous involution of time reversibility of the equations (1)-(2), 3) oscillations of the globally analytic pendulum near "vertical" equilibrium (its canonical equilibrium modulo the time reversibility of equations (1)-(2)), 4) canonical periods of the universal screw motion in Euclidean 3d-space, 5) canonical periods of the 3d-Klein bottle (canonical functional generalization of classic 2d-Klein bottle, see [1]).

Interpretive representations of the general solution

Formulas (3)-(5) represent the general solution of equations (1)-(2), having the following interpretations:

- the canonical multivalued derived continuous monodromic dynamics of the universal accompanying tetrahedron for analytic tops;

- the universal canonical 1-connected analytic structure on 3d-sphere $\mathbb{S}^3$;

- the universal canonical screw motion in three-dimensional Euclidean space;

- the phase flow of the canonical analytic pendulum over $\mathbb{C}\cup\infty$, which is the canonical analytic continuation of the dynamics of the classical pendulum into its vertical equilibrium corresponding to $s, t = \infty$, $t \in \mathbb{R}, s \in \mathbb{C}$.

Pendulum-oscillator realizations of the general solution:

- a pure real realization - the canonical analytical pendulum: oscillations of a classical pendulum about its analytical equilibrium - an equivariant gluing of the lower and upper equilibria at $\mathbb{C}\cup\infty$ by means of the involution of time reversibility of equations (1)-(2)(abbreviated as CAP);

- a pure imaginary realization - the canonical purely imaginary oscillator: oscillations of the canonical globally analytic extension of the classical harmonic oscillator (abbreviated as CAO);

- a mixed (correct diagonal quaternionic) realization - globally analytic spherical pendulum, globally analytic Whitney pendulum;

- the physical realization (represents the physical meaning of the analytic continuation of the dynamics of a classical pendulum at $s = \infty$) - analytically relativistic pendulum, analytically relativistic oscillator.

The paradoxical characteristics of analytically relativistic pendulum/oscillator are as follows: length = speed of light in vacuum, equilibrium period (!) = 281 (the rank of continuous central symmetry mapping in space $\mathbb{E}^3$), oscillation period = e^{281}, oscillation frequency = Planck's constant (special calculation). Scalar invariants 281, e^{281} and their connections with the QFT constants, including the noted one, were discovered by Ya.V.Ryazantsev.

Additional interpretations of the general solution:

- dynamics of the universal axis of rotation of analytic tops representing an operation in the group of spectral data of the inertia ellipsoid of the general top (General Top = GT) having arbitrary analytical parameters $(I, \overrightarrow{c})$;

- the dynamics of a 3d-spherical pendulum/oscillator having a 3d-sphere $\mathbb{S}^3(\mathbb{C})$ in the configuration space;

- the dynamics of the kinetic moment vector $\overrightarrow{M}(t)$ of heavy solids relative to the globally analytic Euler angles $\varphi_{an}=t,\ \psi_{an}=q,\ \theta_{an}=s$, where

- t (coordinate on the pendulum motion of the GT axis; on the 1d-layer $\mathbb{S}^3_{Hopf,an}(\mathbb{C})$);

- q (coordinate on the GT precession motion; based on $\mathbb{S}^3_{Hopf,an}(\mathbb{C})$);

- s (coordinate on the nutational motion of the GT axis, on the space of the bundle $\mathbb{S}^3_{Hopf,an}(\mathbb{C})$),

where

- $\mathbb{S}^3_{Hopf,an}(\mathbb{C})$ is equivariant $\mathbb{C}$-analytic Hopf bundle of the 3d-sphere $\mathbb{S}^3$;

- s is also "the flip-flop coordinate of the Dzhanibekov effect", where the vector $\overrightarrow{M}(t)_{general}$ plays the role of the universal Galois axis (see concrete example in [4]).

Comment on notation (see [1]). The function $\Delta_{12}(q)$ is the only parabolic form of weight 12 relative to the group $SL_2(\mathbb{Z})$:

$$\Delta_{12}(q)=q\prod_{n=1}^{\infty}(1-q^n)^{24},\quad q=e^{2\pi iz},\quad z\in\mathbb{H}^+$$

and the modularity (automorphicity) condition is satisfied:

$$\Delta_{12}(q)\left(\frac{az+b}{cz+d}\right)=(cz+d)^{12}\Delta_{12}(q),$$

where

1. $a,b,c,d\in\mathbb{Z}$ and $ad-bc=1$, the equality defining the function $\Delta_{12}(q)$ as a modular form of weight 12;
2. $\zeta(s,\Delta_{12}(q))$ is the zeta-function of the form $\Delta_{12}(q)$, defined below;
3. $PGL_2(\mathbb{Q}(s/t))$ - a functional extension of the group $PGL_2(\mathbb{Q})$ - the group of matrices of projective linear transformations of the affine plane over the field $\mathbb{Q}$;
4. $\zeta(s,\Delta_{12}(q))=(2\pi)^{-s}\Gamma(s+11/2)L_\Delta(s)$,
   where there is an additive representation
   $$L_\Delta(s)=L(s,\ \Delta_{12}(q))=\sum_{i=1}^{\infty}\frac{\tau(n)/n^{11/2}}{n^s}$$
   and accordingly, the multiplicative representation:
   $$L_\Delta(s)=L(s,\ \Delta_{12}(q))=\prod_p(1-\frac{\tau(p)/p^{11/2}}{p^s}+\frac{1}{p^{2s}}),$$

where p runs over the whole set of primes.

Remark 1. The set of solutions of the transcendental equation

$$\{\zeta(s,\ \Delta_{12}(q))=0\}$$

presents:

1. the absolute of the canonical functional continuous extension of the Lobachevsky 3d-space over $\mathbb{C}$;
2. standard 3d-sphere $\mathbb{S}^3(\mathbb{C})$ (globally with one map over $\mathbb{C}\cup\infty$);
3. the set of axes of rotation of the trivial (ball) top.

Remark 2. The form $\zeta(s, \Delta_{12}(q) \otimes_{SL_2(\mathbb{Z})} PGL_2(\mathbb{Q}(s/t)))$ represents the equivariant continuous correction (triangulation) of the form $\Delta_{12}(q)$ and eventually coincides with the function $\zeta(s, \Delta_{12}(q))$.
The general solution $\overrightarrow{M}(t)_{general}$ of equations (1)-(2) in this context can be considered as the canonical global (over $C\cup\infty$) analytical correction of the parabolic form $\Delta_{12}(q)$.

General integral of the Euler-Poisson equations

Due to the quaternionic realization, it is possible not only to predict the existence of the general integral of equations (1)-(2), but also to write out its explicit form:

$$\exp F = \left|(\omega_1 + i\omega_2 + j\omega_3)^2 + (\gamma_1 + i\gamma_2 + j\gamma_3)\right)|^2$$

with the relations (boundary conditions): $i^2 = j^2 = -1, i + j = ij$.

The invariant $\exp F$ represents: 1) relations in the monodromy group of the universal trihedron accompanying the analytic tops, 2) CAP equilibrium condition, 3) CAO boundary point condition; i - the generator of pendulum (even) oscillations, j - the generator of rotational (odd) oscillations.

The invariant $\exp F$ has the meaning of 1) the potential of canonical conjugation in the canonical equivariant analytic extension of quaternions, 2) the Hamiltonian of the canonical rectilinear flow on the 3d-Klein bottle (this is a functional manifold: 1) the canonical global analytic parameter on the 3d-sphere $\mathbb{S}^3$ (similar to the parameter e^{it} on the circle $\mathbb{S}^1$), 2) the central generating section of the group flow of big circles on the 3d-sphere $\mathbb{S}^3$, see also [1]).

Detailing the structure of the general solution of the Euler-Poisson equations

Theorem 3. The general solution of equations (1, 2) is represented in the form:

$$\overrightarrow{M}(t, t_0)_{general} = \exp\left(\zeta\left(\mathrm{s}, \Delta_{12}(q)\right) \cdot \zeta\left(\mathrm{s}, \Delta_{12}(q)\right) = 0\right).$$

Theorem 4. The general solution of equations (1)-(2) is a group functional CW-complex defined on the set of their particular solutions, having a canonical representation:

$$\overrightarrow{M}(s)_{partial} = (\overrightarrow{M}(s)_{general})_{CW} = \exp\left(\zeta(s, E/\mathbb{Q})\overline{\zeta_0(s, E/\mathbb{Q})}\right),$$

where

1. $E/\mathbb{Q}$ runs over the set of elliptic curves over the field of rational numbers $\mathbb{Q}$;
2. $\zeta(s, E/\mathbb{Q})$ is the zeta function of the elliptic curve $E/\mathbb{Q}$;

the components of the vector $\overrightarrow{\zeta_0(s, E/\mathbb{Q})}$ are embedded in the orbit of the group law on the curve $E/\mathbb{Q}$; they determine the unramified group discretization (vector-valued grading) of the function $\zeta(s, E/\mathbb{Q})$ and have the form:

$$\overrightarrow{\zeta_0\left(s, E\ /\mathbb{Q}\right)} =$$

$$= \left(\left(\zeta\left(s_1, E/\mathbb{Q}\right) = 0\right), \left(\zeta\left(s_2, E/\mathbb{Q}\right) = 0\right), \left(\zeta\left(s_3, E/\mathbb{Q}\right) = 0\right)\right),$$

where the vector $\overrightarrow{\zeta_0\left(s, E\ /\mathbb{Q}\right)}$ is a vector consisting of three consecutive nontrivial zeros of the function $\zeta(s, E/\mathbb{Q})$ (its zeros with nonzero imaginary part).

Corollary 1. The functions $\exp(\zeta(s, E/\mathbb{Q}))$ represent:

1. an ordered (according to the isogenicity classes of the $E/\mathbb{Q}$ curves) set of equivalence classes of equivariant analytical parabolic modular forms with respect to the adjoint group $Ad\ PGL_2(\mathbb{Q})$ - the group of classes of automorphisms of the coefficients of spectral curves $E/\mathbb{Q}$;
2. classes of the canonical $\mathbb{C}$-analytic flow of big circles on the 3d-sphere $\mathbb{S}^3$ - classes of nonequivalent $\mathbb{R}$-analytic 3d-spheres;
3. canonical cycles of the general solution $\overrightarrow{M}(s)_{general}$;
4. potentials of phase flows of integrable cases of equations (1, 2);
5. vibration modes of a 3d-spherical pendulum/oscillator.

Corollary 2. The set of initial conditions of equations (1, 2) corresponding to $t_0 \in \mathbb{R}$ after renormalization by the mapping of the analytic continuation $t \rightarrow t/\mathbb{Z}_2^{E-P}(t)$, where $\mathbb{Z}_2^{E-P}(t)$ is the time-reversibility involution equations (1,2), is a set of ordered vectors of the following form:

$$\overrightarrow{\gamma}\left(t_0\right) = \left\{\zeta\left(\left(s_{1,*}, E/\mathbb{Q}\right) = 0\right), \zeta\left(\left(s_{2,*}, E/\mathbb{Q}\right) = 0\right), \zeta\left(\left(s_{3,*}, E/\mathbb{Q}\right) = 0\right)\right\},$$

$$\overrightarrow{\omega}\left(t_0\right) = \left\{\zeta\left(\left(s_1^*, E/\mathbb{Q}\right) = 0\right), \zeta\left(\left(s_2^*, E/\mathbb{Q}\right) = 0\right), \zeta\left(\left(s_3^*, E/\mathbb{Q}\right) = 0\right)\right\},$$

where

1. $s_{1,*}, s_{2,*}, s_{3,*}$ - arbitrary sequential trivial zeros of the corresponding zeta functions - zeros with the imaginary part zero;
2. s_1^*, s_2^*, s_3^* are arbitrary consecutive non-trivial zeros of the corresponding zeta functions - zeros with a nonzero imaginary part.

The order on the set of indicated zeros is determined correctly:

1. trivial zeros are lying on the real line $\mathbb{R}$ in the complex plane $\mathbb{C}$ (on the GT axis);
2. nontrivial zeros appear to lie on the straight line $\frac{1}{2} + it$ (the "critical line") in the complex plane $\mathbb{C}$ (on the GT border).

Ordered vectors $\overrightarrow{\gamma}(t_0)$, $\overrightarrow{\omega}\left(t_0\right)$:

1. canonically represent the space of initial conditions for equations (1, 2) considered in analytic reversible time $t/\mathbb{Z}_2^{E-P}(t)$ (or - simply in analytical time);

2. have the mechanical meaning of the canonical domain for determining the set of kinetic moment axises (generators of vertical equilibrium of analytic tops - canonical equivariant 3d-analogues of vertical equilibrium of a classical mathematical pendulum);
3. ordered (mod 3) trivial zeros represent successive instantaneous positions of trihedrons accompanying tops;
4. ordered (mod 3) nontrivial zeros represent the successive instantaneous positions of the tetrahedra accompanying the tops.

Theorem 5. The set of classes of particular solutions (integrable cases) represents a group isomorphic to global (over $\mathbb{C} \cup \infty$) analysis of a Gauss group of roots of the 17th degree from unity constructed by a compass and a ruler on the plane, and implements its canonical 3d-analogue:

1. the orbits of the 3d-ruler correspond to the orbits of the vectors $\overrightarrow{\gamma}(t/\mathbb{Z}_2^{E-P}(t))$;
2. the orbits of the 3d-compass correspond to the orbits of the vectors $\overrightarrow{\omega}(t/\mathbb{Z}_2^{E-P}(t))$,

where the group operation has potential - an additional integral F.

Idea of proof: the Euler-Poisson equations are a local form of the canonical global self-duality of the sphere $\mathbb{S}^3$

The key steps in the proof of the above statements are to identify and to coordinatizate the canonical global simply connected analytic structure on the 3d-sphere $\mathbb{S}^3$ (using a global metric F).

The geometric, mechanical and arithmetic meaning of such coordinatization is the introduction of canonical coordinates on the monodromic self-duality motion of the universal accompanying tetrahedron for analytic tops (where the universal accompanying γ-tetrahedron represents "the universal modular curve over $\mathbb{Q}$" and the universal ccompanying ω-tetrahedron represents the universal elliptic curve over $\mathbb{Q}$").

This is precisely the essence of the nonclassical countable structure of the resulting general solution of the Euler-Poisson equations.

This model also provides a correct (equivariant) theorem on the smooth dependence of the solution of equations (1)-(2) from the initial conditions.

The above analytical structure is the canonical group structure of a big circle flow on 3d-sphere $\mathbb{S}^3$ and represents an equivariant functional analytic extension of the group $SO(3)$.

Its differential represents 1) the original equations (1) - (2) (over $\mathbb{R}$), 2) the general analytic perturbation theory of equation (1)-(2) (over $\mathbb{C}$).

Generalized Dzhanibekov effect, massive gravitons, Shafarevich-Tate groups, modularity of elliptic curves over $\mathbb{Q}$, Langlands correspondence

Here are some aspects of the duality "analytic mechanics - analytic arithmetic".

1. The classical Dzhanibekov effect ([4]) is a particular solution of equations (1)-(2) over $t = \mathbb{C}$ representing the minimal oscillation mode (minimal "action-angle" variables) of CAP (see [1]).

Correctness of modeling the Dzhanibekov effect by equations (1) - (2): Due to the equivalence of 1) the CAP and CAO models, 2) the CAO model and the Aksenov-Demin-Grebenikov model of the Earth's gravitational potential (see [1]), the general solution of equations (1)-(2) has a celestial-mechanical (relativistic mechanical) meaning of a mathematical model of:

- the Earth's gravitational potential (over real time);

- gravitational potential of the Earth-Moon system (over complex time) - as a general analytical perturbation of the real solution.

Conjecture. The generalized Dzhanibekov effect represents the classes of solutions of equations (1)-(2) over $t = \mathbb{C}$ - the finite (finally) spectrum of modes of oscillation of the pendulum (types of "action-angle" variables).

2. Nut in Dzhanibekov's experiment can be interpreted as a "massive graviton": zero inertial mass (because weightlessness); spin is equal to 2 (Euler's characteristic of the Poisson sphere - the configuration space of equations (1)-(2)); in a state of relative rest the nut moves with a speed of light (the speed of a relativistic pendulum).

In this context all artificial satellites of the "Earth-Moon" system under the action of external moments have the meaning of "massive gravitons".

Note that for the "standard nut" (with the nominal width equal to 1), within the framework of the used theoretical model, the half-period of the nut "flip-flop" is calculated: it is equal to 42 (this is the amplitude of odd oscillations of the pendulum, equal to the Euler characteristic of the functional analytical sphere $\mathbb{S}^3$ - sphere with canonical analytic big circle flow).

3. **Conjecture.** The spectrum of types of even oscillation modes of CAP is represented by the Shafarevich-Tate groups of elliptic curves over $\mathbb{Q}$ (see the definition in [6]).

4. The modularity property of elliptic curves over $\mathbb{Q}$ represents 1) the trivial mode of oscillation of CAP, 2) the canonical global continuous parameterization on the 3d-sphere $\mathbb{S}^3$, 3) the phase flow of a top with a unit inertia tensor.

The generalized Dzhanibekov effect represents the derived modular parametrization of eliiptic curves over $\mathbb{Q}$ - the canonical multivalued global simply connected analytic parametrization of the 3d-sphere $\mathbb{S}^3$.

We formulate a statement that describes the connection between mechanics and “functional arithmetic” described above, in the context of the modern apparatus of theoretical physics — the Langlands program (see the popular exposition in [5]).

5. **Conjecture (The mechanical meaning of the Langlands correspondence).** The Galois axes, which have the meaning of generators of the mapping of the kinetic momentum $\overrightarrow{M}(t)_{general}$ (see Th.2), represent the orbits of the equivariant Langlands duality for the group $SO(3,\mathbb{R})$ - this is the set (sheave) of eigensections of the canonical *analytic* double-covering of the sphere $\mathbb{S}^3(\mathbb{C})$. The Langlands dual group to $SO(3,\mathbb{R})$ is the functional analytic simple exceptional Lie group $E_8^{an}(\mathbb{C})$ (see [1]), which represents the orbit of the canonical monodromy of the canonical global (over $\mathbb{C}\cup\infty$) duality of the tangent and tangent sheaves to sphere $\mathbb{S}^3(\mathbb{C})$ with potential F.

Conclusion

The relationship between classical mechanics, relativistic physics and transcendental (modular) arithmetic revealed using the Euler-Poisson equations as an example opens up the possibility of using the apparatus of "modular mathematics" in classical mechanics and quantum field theory.

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