Riemann Hypothesis Analogue for Krawtchouk and Discrete Chebyshev Polynomials

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Abstract. We prove that the real parts of all complex zeros of the Krawtchouk polynomials as well as of the Discrete Chebyshev Polynomials of order $N = -1$ are equal to $-\frac{1}{2}$.

1. Introduction

For any family $\mathcal{F} = \{f_i\}_{i \in I}$ of one-variable complex-valued functions defined on a subset of $\mathbb{C}$, let $Z(\mathcal{F})$ be a set of all complex zeros of all functions in $\mathcal{F}$. For any $a \in \mathbb{R}$, $\varrho(a)$ stands for the vertical line $Re(z) = a$ in the complex plane.

For the Riemann zeta-function $\zeta(s)$, the famous Riemann hypothesis (RH) claims that $Z(\zeta) \subset \varrho(\frac{1}{2})$, with the trivial zeros $-2\mathbb{N}$ excluded. Analogously, we say that RH is valid for a family $\mathcal{F}$ (or that family $\mathcal{F}$ has the RH-property) with parameter a if $Z(\mathcal{F}) \subset \varrho(a)$. We will show that RH is valid with $a = -\frac{1}{2}$ for the families of the Krawtchouk polynomials $K_r^{(N)}$ and Discrete Chebyshev polynomials $D_r^{(N)}$ when $N = -1$. We will denote these families $\mathfrak{Kr}^{(-1)}$ and $\mathfrak{DCh}^{(-1)}$ respectively.

2. Preliminaries

2.1. On Krawtchouk polynomials

For general information about Krawtchouk polynomials, we refer to [1, 2], but here underline only two important facts:

1. The generating function of the Krawtchouk polynomials of order N :

$$(1+t)^{N-z}(1-t)^z = \sum_{r=0}^{\infty} K_r^{(N)}(z) t^r, \qquad (z \in \mathbb{C}) \tag{1}$$

2. The recurrent relation for the Krawtchouk polynomials of order N :

$$(r+1)K_{r+1}^{(N)}(z) = (N-2z)K_r^{(N)}(z) - (N-r+1)K_{r-1}^{(N)}(z), \quad r \geq 1 \tag{2}$$

with the initial terms $K_0^{(N)}(z) = 1$, $K_1^{(N)}(z) = N - 2z$.

2.2. On Discrete Chebyshev polynomials

For the definition and properties of Discrete Chebyshev polynomials $D_r^{(N)}$, we refer to [2], and here point out only that

1. A closed form of their generating function is known [3] .

2. The recurrent relation for the Discrete Chebyshev polynomials of order N :

$$r^2 D_r^{(N)}(z) = (2r-1)(N-2z)D_{r-1}^{(N)}(z) - (N+r)(N-r+2)D_{r-2}^{(N)}(z) \quad ,r \geq 2 \tag{3}$$

with the initial terms $D_0^{(N)}(z) = 1,\ D_1^{(N)}(z) = N - 2z$.

2.3. On tridiagonal matrices

Tridiagonal matrices are treated more generally in [5, 6], but here we want to point out the following:

1. A tridiagonal $(n \times n)$-matrix is of form

$$\mathbf{A} = \begin{pmatrix} m_1 & u_1 & 0 & 0 & 0 & \cdots & 0 \\ l_1 & m_2 & u_2 & 0 & 0 & \cdots & 0 \\ 0 & l_2 & m_3 & u_3 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & m_{n-1} & u_{n-1} \\ 0 & 0 & 0 & 0 & \cdots & l_{n-1} & m_n \end{pmatrix}, \tag{4}$$

which is more compactly written as

$$\mathbf{A} = \begin{pmatrix} * & u_1 & u_2 & \ldots & u_{n-1} \\ m_1 & m_2 & \ldots & m_{n-1} & m_n \\ l_1 & l_2 & \ldots & l_{n-1} & * \end{pmatrix} \tag{5}$$

or even more compactly as $\mathbf{A} = [\mathbf{u}, \mathbf{m}, \mathbf{l}]$, where $\mathbf{m} = (m_1, m_2, \ldots, m_{n-1}, m_n)$, $\mathbf{u} = (u_1, u_2, \ldots, u_{n-1})$, $\mathbf{l} = (l_1, l_2, \ldots, l_{n-1})$ represent the main diagonal and the upper (resp. low) subgiagonals in $\mathbf{A}$.

2. Determinant $|\mathbf{A}|_{(n)}$ can be found [5, 6] by recurrent formulas

$$|\mathbf{A}|_{(k)} = m_k|\mathbf{A}|_{(k-1)} - u_{k-1}l_{k-1}|\mathbf{A}|_{(k-2)}, \tag{6}$$

$1 \leq k \leq n$, with $|\mathbf{A}|_{(-1)} = 0, |\mathbf{A}|_{(0)} = 1$.

3. Jacobi theorem on tridiagonal matrices [5, 6]

Theorem 1 (Jacobi). *If in a (real) tridiagonal matrix* $\mathbf{A} = [\mathbf{u}, \mathbf{m}, \mathbf{l}]$ *has* $\mathbf{m} = \mathbf{0}$ *and* $u_k l_k < 0, \forall k, 1 \leq k \leq n-1$ *then* $\mathbf{A}$ *is diagonally similar to a skew-symmetric matrix* $\mathbf{S} = \mathbf{D}^{-1}\mathbf{A}\mathbf{D}$, *where*

$$\mathbf{D} = \mathbf{diag}(d_k),\ d_1 = 1, d_k = \operatorname{sgn}(u_k)\sqrt{\prod_{i=1}^{i=k-1} \operatorname{abs}\Big(\frac{l_i}{u_i}\Big)}\ ,\ 2 \leq k \leq n-1. \tag{7}$$

3. Determinant form for the Krawtchouk and Chebyshev polynomials

Theorem 2. 1.

$$K_k^{(N)}(z) = \frac{(-1)^k}{k!}\Delta_{N,k}^{(Kr)}(z), \tag{8}$$

where

$$\Delta_{N,k}^{(Kr)}(z) = \det \begin{pmatrix} 2z-N & 1 & 0 & 0 & 0 & \cdots & 0 \\ N-2 & 2z-N & 2 & 0 & 0 & \cdots & \\ 0 & N-3 & 2z-N & 3 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & N-(k-1) & \cdots & 2z-N & k-1 \\ 0 & 0 & 0 & 0 & \cdots & N & 2z-N \end{pmatrix}_{k \times k}, \ k \geq 1. \tag{9}$$

i. e, the diagonals of this tridiagonal matrix are as follows :

$$\mathbf{u} = (j)_{1 \leq j \leq k-1},\ \mathbf{m} = (2z-N)_{1 \leq j \leq k},\ \mathbf{l} = (N - \operatorname{mod}(j+1, k))_{1 \leq j \leq k-1} \quad ,k \geq 2 \tag{10}$$

2.
$$D_k^{(N)}(z) = \frac{1}{(k+1)(k!)^3}\Delta_{N,k}^{(D)}(z), \tag{11}$$

where

$$\Delta_{N,k}^{(D)}(z) = \det\begin{pmatrix} 2(N-2z) & N & 0 & 0 & 0 & \dots & 0 \\ 6(N+2) & 9(N-2z) & 4(N-1) & 0 & 0 & \dots & 0 \\ 0 & 12(N+3) & 20(N-2z) & 9(N-2) & 0 & \dots & 0 \\ 0 & 0 & 20(N+4) & 35(N-2z) & 16(N-3) & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \end{pmatrix}_{k\times k} \tag{12}$$

i. e, the diagonals of this tridiagonal matrix are as follows :

$$\mathbf{u} = (j^2(N-j+1))_{1\le j\le(k-1)}, \mathbf{m} = (N-2z)((j+1)(2j-1))_{1\le j\le k},\ \mathbf{l} = ((j+1)(j+2)(N+1+j))_{1\le j\le(k-1)} \tag{13}$$

Proof. 1. For k=0 and k=1 the equality 9 is evident. For $r \ge 2$ let for a time $a_r = \frac{(-1)^r}{r!}\Delta_{N,r}^{(Kr)}(z)$ where we omitted variable z. Then applying to determinant $\Delta_{N,r}^{(Kr)}(z)$ in a_r formulae 6 we get after some elementary calculations the recurrent relation for these polynomials: $(r+1)a_{r+1} = (N-2z)a_r - (N-r+1)a_{r-1},\ \ r \ge 1$. This is exactly equation (2). Hence $a_k = K_k^{(N)}(z)$ for all k.

2. The proof is analogous to the previous one with $a_k = \Delta_{N,k}^{(D)}$ and recurrence 3.

□

4. Analogue of RH for the Krawtchouk and Discrete Chebyshev Polynomials

Theorem 3. 1. $Z(\mathfrak{Kr}^{(-1)}) \subset \varrho(-\frac{1}{2})$.

2. $Z(\mathfrak{DCh}^{(-1)}) \subset \varrho(-\frac{1}{2})$.

Proof. 1. For $N = -1$ the tridiagonal matrix in equality 9 becomes:

$$\Delta_{-1,k}^{(K)}(z) = \det\begin{pmatrix} 2z+1 & 1 & 0 & 0 & 0 & 0 & \dots & 0 \\ -3 & 2z+1 & 2 & 0 & 0 & 0 & \dots & 0 \\ 0 & -4 & 2z+1 & 3 & 0 & 0 & \dots & 0 \\ 0 & 0 & -5 & 2z+1 & 4 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \end{pmatrix}_{k\times k},\ k \ge 1. \tag{14}$$

Since $Z(K_k^{(N)}(z)) = Z(\Delta_{N,k}^{(K)}(z))$ we are interested in the roots of equation

$$\Delta_{-1,k}^{(K)}(z) = 0 \tag{15}$$

which is equivalent to

$$\det\begin{pmatrix} -w & 1 & 0 & 0 & 0 & 0 & \dots & 0 \\ -3 & -w & 2 & 0 & 0 & 0 & \dots & 0 \\ 0 & -4 & -w & 3 & 0 & 0 & \dots & 0 \\ 0 & 0 & -5 & -w & 4 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \end{pmatrix} = 0 \tag{16}$$

with a new variable w so that $z = \frac{-1-w}{2}$.

It is easily seen that the roots of 16 are exactly the eigenvalues of a tridiagonal matrix with zero-(main)diagonal which due to Jacobi theorem is similar to some skew-symmetric matrix. Hence all the values of w being the eigenvalues of a skew-symmetric matrix are pure imaginary and hence the real parts of all values of z are equal to $-\frac{1}{2}$, i. e. $Z(K_k^{(N)}(z)) = Z(\Delta_{N,k}^{(K)}(z) \subset \varrho(-\frac{1}{2})$

2. For $N=-1$ the tridiagonal matrix in equality (11) becomes:

$$\Delta_{-1,k}^{(D)}(z) = \det \begin{pmatrix} 2(-1-2z) & -1 & 0 & 0 & 0 & \cdots & 0 \\ 6(-1+2) & 9(-1-2z) & 4(-1-1) & 0 & 0 & \cdots & 0 \\ 0 & 12(-1+3) & 20(-1-2z) & 9(-1-2) & 0 & \cdots & 0 \\ 0 & 0 & 20(-1+4) & 35(-1-2z) & 16(-1-3) & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix}_{k\times k}, \tag{17}$$

so here similary to the previous item of the proof, we get (after some elementary calculations) equation

$$\det \begin{pmatrix} -w & 1/9 & 0 & 0 & 0 & 0 & \dots & 0 \\ -6/2 & -w & 8/20 & 0 & 0 & 0 & \dots & 0 \\ 0 & -24/9 & -w & 27/35 & 0 & 0 & \dots & 0 \\ 0 & 0 & -60/20 & -w & 64/54 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \end{pmatrix} = 0 \tag{18}$$

where again $z=\frac{-1-w}{2}$. Equation 18 implies that the values w are the eigenvalues of a matrix similar to a skew-symmetric matrix, hence all these values are pure imaginary and hence the real part of variable z is always equal to $-\frac{1}{2}$.

□

Conclusion

We have shown that a family of mathematically interesting polynomials satisfy the RH property. Analogous results have been presented earlier (eg. [4]), but our methods are far more straightforward and hence of larger interest.

References

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