Riemann Hypothesis Analogue for Krawtchouk and Discrete Chebyshev Polynomials

Nikita Gogin and Mika Hirvensalo

Abstract. We prove that the real parts of all complex zeros of the Krawtchouk polynomials as well as of the Discrete Chebyshev Polynomials of order $N = -1$ are equal to $-\frac{1}{2}$.

1. Introduction

For any family $\mathcal{F} = \{f_i\}_{i \in I}$ of one-variable complex-valued functions defined on a subset of $\mathbb{C}$, let $Z(\mathcal{F})$ be a set of all complex zeros of all functions in $\mathcal{F}$. For any $a \in \mathbb{R}$, $\varphi(a)$ stands for the vertical line $Re(z) = a$ in the complex plane.

For the Riemann zeta-function $\zeta(s)$, the famous Riemann hypothesis (RH) claims that $Z(\zeta) \subset \varphi(\frac{1}{2})$, with the trivial zeros $-2N$ excluded. Analogously, we say that RH is valid for a family $\mathcal{F}$ (or that family $\mathcal{F}$ has the RH-property) with parameter $a$ if $Z(\mathcal{F}) \subset \varphi(a)$. We will show that RH is valid with $a = -\frac{1}{2}$ for the families of the Krawtchouk polynomials $K_r^{(N)}$ and Discrete Chebyshev polynomials $D_r^{(N)}$ when $N = -1$. We will denote these families $\mathcal{K}(−1)$ and $\mathcal{DCh}(−1)$ respectively.

2. Preliminaries

2.1. On Krawtchouk polynomials

For general information about Krawtchouk polynomials, we refer to [1, 2], but here underline only two important facts:

1. The generating function of the Krawtchouk polynomials of order $N$:
\begin{equation}
(1 + t)^{N-z} (1 - t)^z = \sum_{r=0}^{\infty} K_r^{(N)}(z) t^r, \quad (z \in \mathbb{C})
\end{equation}

2. The recurrent relation for the Krawtchouk polynomials of order $N$:
\begin{equation}
(r + 1)K_{r+1}^{(N)}(z) = (N - 2z)K_r^{(N)}(z) - (N - r + 1)K_{r-1}^{(N)}(z), \quad r \geq 1
\end{equation}

with the initial terms $K_0^{(N)}(z) = 1$, $K_1^{(N)}(z) = N - 2z$.

2.2. On Discrete Chebyshev polynomials

For the definition and properties of Discrete Chebyshev polynomials $D_r^{(N)}$, we refer to [2], and here point out only that

1. A closed form of their generating function is known [3].
2. The recurrent relation for the Discrete Chebyshev polynomials of order $N$:

$$r^2 D_r^{(N)}(z) = (2r - 1)(N - 2z)D_{r-1}^{(N)}(z) - (N + r)(N - r + 2)D_{r-2}^{(N)}(z), \quad r \geq 2 \tag{3}$$

with the initial terms $D_0^{(N)}(z) = 1, D_1^{(N)}(z) = N - 2z$.

2.3. On tridiagonal matrices

Tridiagonal matrices are treated more generally in [5, 6], but here we want to point out the following:

1. A tridiagonal $(n \times n)$-matrix is of form

$$A = \begin{pmatrix}
m_1 & u_1 & 0 & 0 & \cdots & 0 \\
l_1 & m_2 & u_2 & 0 & \cdots & 0 \\
0 & l_2 & m_3 & u_3 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & m_{n-1} & u_{n-1} \\
0 & 0 & 0 & \cdots & l_{n-1} & m_n
\end{pmatrix}, \tag{4}$$

which is more compactly written as

$$A = \begin{pmatrix}
* & u_1 & u_2 & \cdots & u_{n-1} \\
m_1 & m_2 & \cdots & m_{n-1} & m_n \\
l_1 & l_2 & \cdots & l_{n-1} & *
\end{pmatrix} \tag{5}$$

or even more compactly as $A = [u, m, l]$, where $m = (m_1, m_2, \ldots, m_{n-1}, m_n)$, $u = (u_1, u_2, \ldots, u_{n-1})$, $l = (l_1, l_2, \ldots, l_{n-1})$ represent the main diagonal and the upper (resp. lower) subdiagonals in $A$.

2. Determinant $|A|_{(n)}$ can be found [5, 6] by recurrent formulas

$$|A|_{(k)} = m_k|A|_{(k-1)} - u_k l_{k-1} |A|_{(k-2)}, \tag{6}$$

$1 \leq k \leq n$, with $|A|_{(-1)} = 0$, $|A|_{(0)} = 1$.

3. Jacobi theorem on tridiagonal matrices [5, 6]

**Theorem 1 (Jacobi).** If in a (real) tridiagonal matrix $A = [u, m, l]$ has $m = 0$ and $u_k l_k < 0, \forall k, 1 \leq k \leq n - 1$ then $A$ is diagonally similar to a skew-symmetric matrix $S = D^{-1} A D$, where

$$D = \text{diag}(d_k), \ d_1 = 1, d_k = \text{sgn}(u_k) \sqrt{\prod_{i=1}^{k-1} \text{abs} \left( \frac{l_i}{u_i} \right)}, \ 2 \leq k \leq n - 1. \tag{7}$$

3. Determinant form for the Krawtchouk and Chebyshev polynomials

**Theorem 2.** 1.

$$K_k^{(N)}(z) = \frac{(-1)^k}{k!} \Delta_k^{(N)}(z), \tag{8}$$

where

$$\Delta_k^{(N)}(z) = \det \begin{pmatrix}
2z - N & 1 & 0 & 0 & \cdots & 0 \\
N - 2 & 2z - N & 2 & 0 & \cdots & 0 \\
0 & N - 3 & 2z - N & 3 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\
0 & 0 & 0 & N - (k - 1) & \cdots & 2z - N \\
0 & 0 & 0 & 0 & \cdots & N - 2z + N
\end{pmatrix}_{k \times k}, \ k \geq 1. \tag{9}$$

i.e., the diagonals of this tridiagonal matrix are as follows:

$$u = (j)_{1 \leq j \leq k-1}, \ m = (2z - N)_{1 \leq j \leq k}, \ l = (N - \text{mod}(j + 1, k))_{1 \leq j \leq k-1}, \ k \geq 2. \tag{10}$$
1. For \( k=0 \) and \( k=1 \) the equality (9) is evident. For \( u = (\pm 1)z \) we are interested in the roots of equation (10), where

\[
\Delta_{N,k}^{(D)}(z) = \det \left( \begin{array}{cccccc}
2(N-2z) & N & 0 & 0 & 0 & \ldots & 0 \\
6(N+2) & 9(N-2z) & 4(N-1) & 0 & 0 & \ldots & 0 \\
0 & 12(N+3) & 20(N-2z) & 9(N-2) & 0 & \ldots & 0 \\
0 & 0 & 20(N+4) & 35(N-2z) & 16(N-3) & \ldots & 0 \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
\end{array} \right)_{k \times k}
\]

is as follows:

\[
u = (j^2(N-j+1))_{1 \leq j \leq (k-1)}, \quad m = (N-2z)(j+1)(2j-1))_{1 \leq j \leq k}, \quad l = ((j+1)(j+2)(N+1+j))_{1 \leq j \leq (k-1)}
\]

Proof. 1. For \( k=0 \) and \( k=1 \) the equality (9) is evident. For \( r \geq 2 \) let for a time \( a_r = \frac{(-1)^{r-1}}{r} \Delta_{N,r}^{(K)}(z) \) where we omitted variable \( z \). Then applying to determinant \( \Delta_{N,r}^{(K)}(z) \) in \( a_r \) formula (8) we get after some elementary calculations the recurrent relation for these polynomials:

\[
(r+1) a_{r+1} = (N-2z)a_r - (N-r+1)a_{r-1}, \quad r \geq 1.
\]

This is exactly equation (12). Hence \( a_k = K_k^{(N)}(z) \) for all \( k \).

2. The proof is analogous to the previous one with \( a_k = \Delta_{N,k}^{(D)} \) and recurrence (3).

\[ \square \]

4. Analogue of RH for the Krawtchouck and Discrete Chebyshev Polynomials

Theorem 3. 1. \( Z(\mathcal{C}(r^{-1})) \subset \varrho(-\frac{1}{2}) \).

2. \( Z(\mathcal{D}(r^{-1})) \subset \varrho(-\frac{1}{2}) \).

Proof. 1. For \( N = -1 \) the tridiagonal matrix in equality (1) becomes:

\[
\Delta_{1,k}^{(K)}(z) = \det \left( \begin{array}{cccccc}
z+1 & 1 & 0 & 0 & 0 & \ldots & 0 \\
-3 & 2z + 1 & 2 & 0 & 0 & \ldots & 0 \\
0 & -4 & 2z + 1 & 3 & 0 & \ldots & 0 \\
0 & 0 & -5 & 2z + 1 & 4 & 0 & \ldots & 0 \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
\end{array} \right)_{k \times k}
\]

Since \( Z(K_k^{(N)}(z)) = Z(\Delta_{1,k}^{(K)}(z)) \) we are interested in the roots of equation

\[
\Delta_{1,k}^{(K)}(z) = 0
\]

which is equivalent to

\[
\det \left( \begin{array}{cccccc}
-w & 1 & 0 & 0 & 0 & \ldots & 0 \\
-3 & -w & 2 & 0 & 0 & \ldots & 0 \\
0 & -4 & -w & 3 & 0 & \ldots & 0 \\
0 & 0 & -5 & -w & 4 & 0 & \ldots & 0 \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
\end{array} \right) = 0
\]

with a new variable \( w \) so that

\[
z = \frac{1-w}{2}
\]

It is easily seen that the roots of (11) are exactly the eigenvalues of a tridiagonal matrix with zero-(main)diagonal which due to Jacobi theorem is similar to some skew-symmetric matrix. Hence all the values of \( w \) being the eigenvalues of a skew-symmetric matrix are pure imaginary and hence the real parts of all values of \( z \) are equal to \( -\frac{1}{2} \). i. e. \( Z(K_k^{(N)}(z)) = Z(\Delta_{1,k}^{(K)}(z)) \subset \varrho(-\frac{1}{2}) \).
2. For $N = -1$ the tridiagonal matrix in equality (11) becomes:

$$
\Delta_{-1,k}^{(D)}(z) = \det \begin{pmatrix}
2(-1 - 2z) & -1 & 0 & 0 & \cdots & 0 \\
6(-1 + 2) & 9(-1 - 2z) & 4(-1 - 1) & 0 & \cdots & 0 \\
0 & 12(-1 + 3) & 20(-1 - 2z) & 9(-1 - 2) & 0 & \cdots & 0 \\
0 & 0 & 20(-1 + 4) & 35(-1 - 2z) & 16(-1 - 3) & \cdots & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots
\end{pmatrix}_{k \times k},
$$

so here similarity to the previous item of the proof, we get (after some elementary calculations) equation

$$
\det \begin{pmatrix}
-w & 1/9 & 0 & 0 & 0 & \cdots & 0 \\
-6/2 & -w & 8/20 & 0 & 0 & \cdots & 0 \\
0 & -24/9 & -w & 27/35 & 0 & \cdots & 0 \\
0 & 0 & -60/20 & -w & 64/54 & 0 & \cdots & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots
\end{pmatrix} = 0 \quad (18)
$$

where again $z = \frac{-1-w}{2}$. Equation (18) implies that the values $w$ are the eigenvalues of a matrix similar to a skew-symmetric matrix, hence all these values are pure imaginary and hence the real part of variable $z$ is always equal to $-\frac{1}{2}$.

\[\square\]

**Conclusion**

We have shown that a family of mathematically interesting polynomials satisfy the RH property. Analogous results have been presented earlier (eg. [4]), but our methods are far more straightforward and hence of larger interest.

**References**


Nikita Gogin

Mika Hirvensalo

Department of Mathematics and Statistics

University of Turku

Turku, Finland

e-mail: mikhirve@utu.fi