Eigenvalues and Eigenvectors for the Composition of Lorentz Boosts in Concise Form

Mikhail Kharinov

Abstract. This paper considers the formal eigenvalue/eigenvector problem for Lorentz transformation $\mathcal{L}$ in the real four-dimensional spacetime $\mathbb{R}^4$. According to the problem statement, it is required to find a quartet of linearly independent eigenvectors for the composition $\mathcal{L} = L_1 L_2$ of the boosts L_1 and L_2. To analytically find the eigenvalues, a fourth-degree polynomial characteristic equation is obtained and solved. The a priori expected concise expressions for the eigenvectors are presented.

Introduction

This work and [1] complete the phase of the study of general Lorentz transformations, begun in [2] and continued in [3]. In [2], a special case of Lorentz boost composition was not considered. In [3], the expression for the eigenvectors of the boost composition turned out to be too cumbersome. The latter disadvantage is overcome in this paper.

The Lorentz transformations $\mathcal{L}$ are defined as a linear homogeneous transformation of the spacetime vectors u, v that preserves the real inner product $(u, \bar{v})$ of one *conjugated* vector $\bar{v} \equiv 2(v, i_0) - v$ by another vector u:

$$(\mathcal{L}\{u\}, \overline{\mathcal{L}\{v\}}) = (u, \bar{v}),$$

where i_0 is the *unit* vector of unit length $\sqrt{(i_0, i_0)} \equiv 1$ along the time axis.

For brevity, only one option of $\pm\mathcal{L}\{u\}$ and $\pm\mathcal{L}\{\bar{u}\}$ is treated.

The transformation $\mathcal{L}$ involves Lorentz boost L as *self-adjoint* transform i.e. in an inner product L is transferred from one vector to another. So, for any u, v $(L\{u\}, v) = (u, L\{v\})$.

The problem is to obtain the quartet of eigenvectors c_k for the transformation $L_1 L_2$:

$$L_1 L_2\{c_k\} = \xi_k c_k \Leftrightarrow L_2\{c_k\} = \xi_k L_1^{-1}\{c_k\}, \tag{1}$$

where ξ_k is the real eigenvalue and the eigenvector serial number k ranges from 0 to 3. For simpler calculations, it is better to treat the equation located in (1) on the right.

1. General statements

To solve the equation (1), the easily provable general considerations are very useful. They are as follows:

1. The desired quartet of eigenvectors is always exist.
2. If ξ is an eigenvalue, then $\frac{1}{\xi}$ is also an eigenvalue.
3. If an eigenvalue ξ is different from 1, then it corresponds to a *lightwise* eigenvector c having a zero pseudo–length: $\xi \neq 1 \Rightarrow (c, \bar{c}) = 0$.

From these statements it is easy to establish without paper calculations that the quartet of eigenvalues consists of two units and a pair of mutually inverse values: 1, 1, ξ and $\frac{1}{\xi}$.

The characteristic equation for ξ is:

$$(\xi - 1)^2(\xi^2 - 2\xi \cosh \chi + 1) = 0, \tag{2}$$

where the scalar parameter χ is defined in accordance with famous cosine rule:

$$\cosh \frac{\chi}{2} = \cosh \frac{\theta_1}{2} \cosh \frac{\theta_2}{2} + (n_1, n_2) \sinh \frac{\theta_1}{2} \sinh \frac{\theta_2}{2} \tag{3}$$

and the scalar parameters θ_1 and θ_2 are the *rapidities*, such that the velocities v_1, v_2 divided by scalar speed of light c are expressed as $\mathrm{v}_1/\mathrm{c} = n_1 \tanh \theta_1$, $\mathrm{v}_2/\mathrm{c} = n_2 \tanh \theta_2$.

Note that (3) refers to the half hyperbolic angles $\frac{\chi}{2}$, $\frac{\theta_1}{2}$ and $\frac{\theta_2}{2}$, while the famous velocity addition is expressed via whole hyperbolic angles θ, θ_1 and θ_2 [4, 5].

From the above and concomitant considerations, we can conclude that the eigenvectors c_0, c_1, c_2, c_3 form a system of *pseudo-orthogonal* vectors, such that $(c_0, \bar{c}_0) = (c_0, \bar{c}_2) = (c_0, \bar{c}_3) = (c_1, \bar{c}_1) = (c_1, \bar{c}_2) = (c_1, \bar{c}_3) = (c_2, \bar{c}_3) = 0$.

2. Eigenvectors

The eigenvectors c_0, c_1, c_2, c_3 for the composition $L_1 L_2$ of Lorentz boosts L_1, L_2 and the corresponding eigenvalues are listed in Table 1.

In Table 1 n_1 and n_2 are the unit *spatial* vectors along the considered intersecting velocities, such that $(n_1, n_1) = (n_2, n_2) = 1$ and $(n_1, i_0) = (n_2, i_0) = 0$. The cross product $[n_1, n_2]$ is directed along the Wigner rotational axis ν [6], so that $[n_1, n_2] = \nu\sqrt{1 - (n_1, n_2)^2}$. The spatial part of the eigenvectors c_0 and c_1 depends

Notation	Eigenvector	Eigenvalue
c_0	$i_0 - d\vert_{\xi=e^{\chi}}$	e^{χ}
c_1	$i_0 - d\vert_{\xi=e^{-\chi}}$	$e^{-\chi}$
c_2	$i_0 - n_1\frac{\coth\frac{\theta_1}{2}+(n_1,n_2)\coth\frac{\theta_2}{2}}{1-(n_1,n_2)^2} + n_2\frac{\coth\frac{\theta_2}{2}+(n_1,n_2)\coth\frac{\theta_1}{2}}{1-(n_1,n_2)^2}$	1
c_3	$[n_1, n_2]$	1

TABLE 1. Eigenvectors for the composition of Lorentz boosts L_1L_2

on the eigenvalue ξ and, up to the sign, coincides with the unit vector d_ξ that is defined as a function of eigenvalue ξ in the form:

$$d_\xi = \frac{n_1\sqrt{\xi}\sinh\frac{\theta_1}{2} + n_2\sinh\frac{\theta_2}{2}}{\sqrt{\xi}\cosh\frac{\theta_1}{2} - \cosh\frac{\theta_2}{2}}. \tag{4}$$

The spatial parts $d|_{\xi=e^{\chi}}$ and $d|_{\xi=e^{-\chi}}$ of the eigenvectors c_0 and c_1 are obtained by substituting into (4) the values $\xi = e^{\chi}$ and $\xi = e^{-\chi}$, respectively.

Thus, in the context of the eigenvalue/eigenvector problem, the composition of Lorentz boosts is as *elementary* as a single Lorentz boost. In both cases, the solution boils down to stretching of one basis eigenvector and reverse decreasing of the second basis eigenvector with the remaining basis eigenvectors unchanged.

A ready-made solution of the eigenvalue/eigenvector problem for the composition of any rotation with a boost, as well as the expressions for representing the composition L_1L_2 of the boosts L_1, L_2 as a composition of the Wigner rotation and boost is given in [2, 3].

Conclusion

The relations (2)–(4) seem perfectly concise and quite simple to be widely presented in reference books to all whom it may concern. But these are missing. The only obstacle to obtaining the above formulae is cumbersome calculations. Two things are important to overcome this obstacle. To simplify the formulae it is useful, firstly, to use hyperbolic geometry, as prescribed in [4, 5], and secondly, to carry out calculations in terms of quaternions [7].

This paper presents a solution to the eigenvalue/eigenvector problem in the vein of [4, 5] in coordinate–free way using the conventional cross product of four-dimensional vectors. In fact, the formulae (2)–(4) turned out to be easier to obtain in terms of the quaternion algebra equipped with quaternionic multiplication [8].

It's remarkable that a modern cross product of vectors is best defined in quaternions and generalized to the case of three arguments [9]. A triple cross product is especially convenient for describing Lorentz transformations, which are represented by a linear combination of orthogonal transforms and are described by triple products of variable vector and constant vector parameters [2, 3].

Among normalized algebras with a multiplicative unit, quaternions expand to octonions. In this case, a cross product of vectors is also generalized to the eight-dimensional case [8, 9, 10, 11]. Along the way, a generalization of Lorentz transformations to eight-dimensional spacetime is anticipated. Probably in the future it will be extremely interesting to generalize the laws of motion from the conditions of their invariance with respect to generalized Lorentz transformations.

The concise representation and description of the Lorentz transformations via eigenvectors may be useful for researchers who will be engaged in the mentioned generalization.

This work was performed within the framework of the budgetary theme 0060-2019-0011 (Fundamentals and technologies of big data for sociocyberphysical systems).

References

[1] M. Kharinov, *The Quartet of Eigenvectors for Quaternionic Lorentz Transformation*, Springer Nature, Adv. Appl. Clifford Algebras, **30**(25), 2020, 20 p.

[2] M. Kharinov, *Sketch on quaternionic Lorentz transformations*, Int. Conf. on Polynomial Computer Algebra (PCA'2019), St.Petersburg, April 15–20, 2019, 71–74.

[3] M. Kharinov, *The Eigenvector Quartet for the Composition of Lorentz Boosts in the Quaternionic Space*, Proc. of the 15-th Int. Conf. on Finsler Extensions of Relativity Theory-2019 (FERT-2019), Moscow, October 24–27, 2019, 195–200.

[4] A.A. Ungar, *Hyperbolic geometry*, Fifteenth International Conference on Geometry, Integrability and Quantization, Varna, Bulgaria, 2013, 259–282.

[5] J.F. Barrett, *Minkovski Space-Time and Hyperbolic Geometry*, MASSEE International Congress on Mathematics MICOM-2015, 2015, 14 pp., available at `https://eprints.soton.ac.uk/397637/2/J\_F\_Barrett\_MICOM\_2015\_2018\_revision\_.pdf`.

[6] C. Møller, *The Theory of Relativity*, Oxford University Press, 1955, 386 pp.

[7] I.L. Kantor, A.S. Solodovnikov, *Hypercomplex numbers: an elementary introduction to algebras*, Springer, 1989, 166 pp.

[8] M. Kharinov, *Product of three octonions*, Springer Nature, Adv. Appl. Clifford Algebras, **29**(1), 2019, 11–26.

[9] T. Dray, C.A. Manogue, *The octonionic eigenvalue problem*, Springer Nature,, Adv. Appl. Clifford Algebras, **8**(2), 1998, 341–364.

[10] Z.K. Silagadze, *Multi-dimensional vector product*, Journal of Physics A: Mathematical and General **35**(23), Institute of Physics Publishing, 2002, 4949–4953.

[11] S. Okubo, *Triple products and Yang–Baxter equation. I. Octonionic and quaternionic triple systems*, Journal of mathematical physics **34**(7), 3273–3291 (1993)

Mikhail Kharinov
St. Petersburg Federal Research Center
of the Russian Academy of Sciences (SPC RAS)
St. Petersburg, Russia
e-mail: `khar@iias.spb.su`