How many roots of a system of random Laurent polynomials are real?

(zeroes of trigonometric polynomials)

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Март, 2021

Expected number of zeroes

Let f be a random real polynomial of degree m. The expected number of its real zeroes asymptotically equals $\frac{2}{\pi}\log m$ (M. Kac. Bull. Amer. Math. Soc. 49, 1943)

Laurent polynomial is a function $f=\sum_{m\in\Lambda} a_m z^m$ on $(\mathbb{C}\setminus 0)^n$, where Λ (the *support* of f) is a finite subset of $\mathbb{Z}^n$ and $z^m = z_1^{m_1}\cdots z_n^{m_n}$. The zero of f in compact subtorus $T^n=\{\mathrm{e}^{\mathrm{i}\theta_1}\times\ldots\times\mathrm{e}^{\mathrm{i}\theta_n}\}$ of the torus $(\mathbb{C}\setminus 0)^n$ is called a *real zero* of f.

Definition 1. Laurent polynomial is called a real Laurent polynomial if its restriction to T^n is a real-valued function.

Lemma 1. Laurent polynomial is real $\Leftrightarrow \forall m\in\mathbb{Z}^n:\ a_m=\overline{a_{-m}}$.
Corollary 1. The support of a real Laurent polynomial is centally symmetric.
Corollary 2. The zero set of a real Laurent polynomial is invariant for the mapping $z\mapsto \bar{z}^{-1}$.

What is the Kac theorem for Laurent polynomial of one variable?
We will find here a phenomenon quite opposite to that of Kac theorem.

AntiKac theorem, 1-dimensional case

Theorem. Let $P(z) = \sum_{1 \le k \le m} a_k z^k + \overline{a_k} z^{-k}$ be a real Laurent polynomial in $\mathbb{C} \setminus 0$ of degree m. Then the expected number of real zeroes is

$$2\sqrt{\frac{m(m+1)}{3}}.$$

Corollary. The expected fraction of real zeroes in the set of all zeroes is

$$\sqrt{\frac{m+1}{3m}}.$$

So the fraction of real zeroes converges to $\sqrt{\frac{1}{3}}$ (> 0.5!).

With differing assumptions for coefficient distribution see
(Jürgen Angst, Federico Dalmao and Guillaume Poly. On the real zeros of random trigonometric polynomials with dependent coefficients. Proc. Amer. Math. Soc. (147:1), 2019, 205–214 (arXiv:1706.01654)

Trigonometric polynomials

If $|z| = 1$ and $a = \alpha + i\beta$ then $az^k + \bar{a}z^{-k} = 2\alpha\cos\langle\theta, k\rangle + 2\beta\sin\langle\theta, k\rangle$. So (from Lemma 1) the restriction $\tilde{f}$ of a real Laurent polynomial $f(z) = \sum_{k\in\Lambda} a_k z^k$ to T^n is a trigonometric polynomial

$$\tilde{f}(\theta) = \alpha_0 + \sum_{k\in\Lambda\setminus 0, \alpha_k, \beta_k \in \mathbb{R}} \alpha_k \cos(k, \theta) + \beta_k \sin(k, \theta),$$

were $\alpha_0 = a_0$, $\alpha_k = \frac{a_k + \overline{a_k}}{2}$, $\beta_k = \frac{a_k - \overline{a_k}}{2i}$.

Lemma 2. The mapping $f \mapsto \tilde{f}$ is an isomorphism of a real vector space of real Laurent polynomials supported in Λ to the space $\mathrm{Trig}(\Lambda)$ of trigonometric polynomials with spectrum Λ.

Expected number of zeroes

We consider the space $\mathrm{Trig}(\Lambda)$ with the metric $L^2(T^n, d\chi)$, where $d\chi$ is the normalized invariant measure in T^n. For $f_1, \ldots, f_n \in \mathrm{Trig}(\Lambda)$ we denote by $N(f_1, \ldots, f_n)$ the number of isolated common zeroes of trigonometric polynomials $f_1, \ldots, f_n$.

Definition. Let S be a unit sphere in $\mathrm{Trig}(\Lambda)$. We say that

$$\mathfrak{M}(\Lambda) = \int_{(s_1,\ldots,s_n)\in(S)^n} N(s_1, \ldots, s_n)\, ds_1 \ldots ds_n$$

is the *average number of roots of a system of random trigonometric polynomials with the support* Λ.

Here we integrate over the orthogonally invariant normalized measure of the sphere S.

Newton ellipsoid, 1

Below we define the centered in 0 ellipsoid $\mathrm{ell}(\Lambda) \subset \mathbb{R}^n$ depending on the support Λ, such that the following is true.

Theorem. (see https://arxiv.org/pdf/2102.00782.pdf and the bibliography therein)

$$\mathfrak{M}(\Lambda) = n!\,\mathrm{vol}(\mathrm{ell}(\Lambda)) \quad (1)$$

Remark 1. In 1-dimensional case (as in 1-dimensional AntiKac Theorem) the ellipsoid $\mathrm{ell}(\Lambda)$ is a centrally symmetric segment $[-a(\Lambda), a(\Lambda)]$, and

$$\mathfrak{M}(\Lambda) = 2a(\Lambda).$$

Remark 2. The mixed version of the Theorem exists, but we do not use it.

Newton ellipsoid, 2

Let $\mathbb{R}^{n*}$ be a space of linear functionals in $\mathbb{R}^n$, and let $\mathbb{Z}^n \subset \mathbb{R}^{n*}$ be an integer lattice, $N = \#(\Lambda)$. Consider the quadratic form

$$F_\Lambda(\xi) = \frac{1}{N} \sum_{m \in \Lambda} m^2(\xi) \tag{2}$$

in $\mathbb{R}^n$. This form is non-negative, and the function

$$h_\Lambda(\xi) = \sqrt{F_\Lambda(\xi)}$$

is convex and positively defined of degree 1. So h_Λ is a support function of some convex body in $\mathbb{R}^{n*}$. This body is an ellipsoid. Denote by $\mathrm{ell}(\Lambda)$. The Theorem is now fully formulated.

Remark 3. If F_Λ is the square of the length of some Euclidean metric in $\mathbb{R}^n$, then $\mathrm{ell}(\Lambda)$ is the unit ball of a dual metric.

Proof of 1-dimensional AntiKac Theorem

If $\Lambda = \{-m, \dots, 0, \dots, m\}$ then, from (2),

$$F_\Lambda(\xi) = \frac{2}{2m+1} \sum_{1 \leq k \leq m} m^2 \xi^2$$

and $h_\Lambda = \sqrt{F_\Lambda}$. In 1-dimensional case $\mathrm{ell}(\Lambda) = [-a, a]$, and, by definition of support function, $a = h_\Lambda(1)$.

As it known $F_\Lambda(1) = \frac{2}{2m+1} \sum_{1 \leq k \leq m} m^2 = \frac{m(m+1)}{3}$, and $a = h_\Lambda(1) = \sqrt{\frac{m(m+1)}{3}}$. From Theorem it follows that (as promised)

$$\mathfrak{M}(\Lambda) = 2\sqrt{\frac{m(m+1)}{3}}$$

and the *expectation of the fraction of real zeroes* equals to $\frac{\mathfrak{M}(\Lambda)}{2m} = \sqrt{\frac{m+1}{3m}}$ and converges to $\frac{1}{\sqrt{3}}$ for $m \to \infty$.

Multidimensional AntiKac Theorem

Let $\Lambda_m = \mathbb{Z}^n \cap B_m$, where B_m is a ball of radius m centered in 0. For real Laurent polynomials $f_1, \dots, f_n$ with support Λ_m denote by $N(f_1, \dots, f_n)$ and by $N_{\mathbb{C}}(f_1, \dots, f_n)$ respectively the numbers of real and complex common zeroes of polynomials $f_1, \dots, f_n$.

Let real_m be an expectation of the random variable $\frac{N(f_1,\dots,f_n)}{N_{\mathbb{C}}(f_1,\dots,f_n)}$, i.e. real_m is a mean value of the fraction of real roots of a random system of real Laurent polynomials with support Λ_m.

AntiKac theorem.

$$\lim_{m\to\infty} \mathrm{real}_m = \left(\frac{\sigma_{n-1}}{\sigma_n}\beta_n\right)^{\frac{n}{2}},$$

where

$$\beta_n = \int_{-1}^{1} x^2(1-x^2)^{\frac{n-1}{2}}\,dx,$$

and σ_k is a volume of k-dimensional unit ball.

The values of β_n

The integral $\beta_n = \int_{-1}^{1} x^2(1-x^2)^{\frac{n-1}{2}}\,dx$ is calculated in elementary functions. Below are its values for $n \le 20$.

If $n = 1$, then $\sqrt{\frac{\sigma_0}{\sigma_1}\beta_1} = \sqrt{\frac{\beta_1}{2}} = \frac{1}{\sqrt{3}}$.

n	1	2	3	4	5	6	7	8	9	10
β_n	$\frac{2}{3}$	$\frac{\pi}{8}$	$\frac{4}{15}$	$\frac{\pi}{16}$	$\frac{16}{105}$	$\frac{5\pi}{128}$	$\frac{32}{315}$	$\frac{7\pi}{256}$	$\frac{256}{3465}$	$\frac{21\pi}{1024}$
n	11	12	13	14	15	16	17	18	19	20
β_n	$\frac{512}{9009}$	$\frac{33\pi}{2048}$	$\frac{4096}{109395}$	$\frac{429\pi}{32768}$	$\frac{2048}{45045}$	$\frac{715\pi}{65536}$	$\frac{65536}{2078505}$	$\frac{2431\pi}{262144}$	$\frac{131072}{4849845}$	$\frac{4199\pi}{524288}$

Proof of AntiKac theorem

Using the Newton ellipsoid theorem (1) and Kushnirenko theorem about the number of common zeros of n Laurent polynomials, we obtain the following statement.

Proposition 1. $\mathrm{real}_m = \frac{\mathrm{vol}(\mathrm{ell}(\Lambda_m))}{\mathrm{vol}(\mathrm{conv}(\Lambda_m))}$

Further, the proof of the theorem splits into the following two simple statements.

Proposition 2. **The sequence of convex bodies $\frac{1}{m}\mathrm{conv}(\Lambda_m)$ converges (in Hausdorff topology) to the unit ball.**

Proposition 3. **The sequence of ellipsoids $\frac{1}{m}\mathrm{ell}(\Lambda_m)$ converges to the ball of radius $\sqrt{\frac{\sigma_{n-1}}{\sigma_n}\beta_n}$, where $\beta_n = \int_{-1}^{1} x^2(1-x^2)^{\frac{n-1}{2}}\,dx$.**

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THANKS FOR ATENTION !