Compact involutive division based on the Janet tree

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Abstract. Gröbner Based on the minimal Gröbner basis G of a monomial ideal $\mathcal{I}$ in the commutative polynomial ring $\mathcal{K}[x_1,\ldots,x_n]$ over a field $\mathcal{K}$ and a total monomial ordering $\succ$, we define another monomial ordering $\succ_G$ such that pairwise involutive partition of variables $\{x_1,\ldots,x_n\}$ for monomials in $\mathcal{I}$ generated by $\succ_G$ yields more compact involutive basis than that generated by $\succ$. In particular, the involutive basis for $\succ_{alex_G}$ and $n \gg 1$ is much more compact then involutive basis for other divisions. We propose a division $\succ_{lex_G}$ based on the Janet tree that is about as compact as $\succ_{alex_G}$, but much more computationally efficient.

1. Involutive division

An involutive division $\mathcal{L}$ is defined on $\mathcal{M} := \{\ x_1^{i_1}\cdots x_n^{i_n} \mid i_j \in \mathbb{N}_{\geq 0}\ \}$ if for any finite $U \subset \mathcal{M}$ and $\forall u \in U$ defined a submonoid $\mathcal{L}(u,U)$ of $\mathcal{M}$ s.t.

1. $w \in \mathcal{L}(u,U),\ v|w \Longrightarrow v \in \mathcal{L}(u,U)$ (partition of variables)
2. $u,v \in U,\ u\mathcal{L}(u,U) \cap v\mathcal{L}(v,U) \neq \emptyset \Longrightarrow u \in v\mathcal{L}(v,U)\ \vee\ v \in u\mathcal{L}(u,U)$ ("unicity" of $\mathcal{L}$-divisor)
3. $v \in U,\ v \in u\mathcal{L}(u,U) \Longrightarrow \mathcal{L}(v,U) \subseteq \mathcal{L}(u,U)$ (transitivity)
4. $V \subseteq U \Longrightarrow \mathcal{L}(u,U) \subseteq \mathcal{L}(u,V)\ \ \forall u \in V$ (filter axiom)

Elements of $\mathcal{L}(u,U)$ are ($\mathcal{L}-$)multiplicative for u.

$$u \in U \quad \Rightarrow \quad \{x_1,\ldots,x_n\} = \underset{\substack{\uparrow \\ \text{multiplicative}}}{M_{\mathcal{L}}(u,U)} \biguplus \underset{\substack{\uparrow \\ \text{nonmultiplicative}}}{NM_{\mathcal{L}}(u,U)}$$

$\boxed{w \in u\mathcal{L}(u,U) \Longleftrightarrow u \mid_{\mathcal{L}} w}$ u is involutive divisor ($\mathcal{L}-$divisor) of w

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$\mathcal{C}_\mathcal{L}(u,U) := u\mathcal{L}(u,U)$ is involutive cone ($\mathcal{L}-$cone) generated by $u \in U$.

2. Involutive bases

Given an ideal $\mathcal{I} \subset \mathcal{K}[x_1,\dots,x_n]$, involutive division $\mathcal{L}$ and monomial order $\succ$, a finite subset $G \subset \mathcal{I}$ is called $(\mathcal{L})-$involutive basis of $\mathcal{I}$ if

$$(\forall f \in \mathcal{I})\ (\exists g \in G)\ [\ \mathrm{lm}(g) \mid_\mathcal{L} \mathrm{lm}(f)\ ]$$

$$\Updownarrow \quad (\text{ for continuous } \mathcal{L}\ )$$

$$(\ \forall f \in G\ )\ (\ \forall x_i \in NM_\mathcal{L}(\mathrm{lm}(f),\mathrm{lm}(G))\ )\ \ [\ NF_\mathcal{L}(\underbrace{x_i \cdot f}, G) = 0\ ]$$

$$\uparrow$$

nonmultiplicative prolongation

$$\underbrace{\mathrm{lm}(g) \mid_\mathcal{L} \mathrm{lm}(f) \Longrightarrow \mathrm{lm}(g) \mid \mathrm{lm}(f)}$$

$$\Downarrow$$

An involutive basis is a Gröbner basis (GB), generally, redundant.

Similarly to a reduced GB a monic minimal involutive basis is unique.

3. Pairwise Construction of Involutive Divisions

All known involutive divisions satisfying the above Axioms 1-3 are pairwise (Gerdt'01), i.e. for any finite set $U \subset \mathbb{M}$ with cardinality $|U| \geq 2$ the set of its $\mathcal{L}$-nonmultiplicative variables is given by

$$(\ \forall u \in U\ )\ \ [\ NM_\mathcal{L}(u,U) = \bigcup_{v \in U\setminus\{u\}} NM_\mathcal{L}(u,\{u,v\})\ ]$$

The pair property provides a regular procedure for construction of a pairwise involutive division called $\sqsupset$-division (Gerdt,Blinkov'11) if it is generated by a total monomial order $\sqsupset$ under the fixed permutation σ on the variables

$$NM_\sqsupset(u,\{u,v\}) := \begin{cases} \text{if } u \sqsupset v \text{ or } (u \sqsubset v \wedge v \mid u) \text{ then } \emptyset \\ \text{else } \{x_{\sigma(i)}\},\ i = \min\{j \mid \deg_{\sigma(j)}(u) < \deg_{\sigma(j)}(v)\} \end{cases}$$

Remark There are $n!$ distinct $\sqsubset$-divisions where n is a number variables.

If $\sqsupset$ is admissible or the negation of admissible, then $\sqsupset$-division is continuous, constructive and Noetherian (Semenov'06, Semenov,Zyuzikov'07-08, Gerdt,Blinkov'11)

Some particular divisions:

- Janet division generated by the lexicographic order $\succ_{\mathrm{lex}}$
- $\succ_{\mathrm{grlex}}$-division generated by the graded lexicographic order:
  $u \succ_{\mathrm{grlex}} v \Longleftrightarrow \deg(u) > deg(v)\ \vee\ deg(u) = deg(v)\ \wedge\ u \succ_{\mathrm{lex}} v$
- $\succ_{\mathrm{alex}}$-division generated by the antigraded lexicographic order:
  $u \succ_{\mathrm{alex}} v \Longleftrightarrow \deg(u) < deg(v)\ \vee\ deg(u) = deg(v)\ \wedge\ u \succ_{\mathrm{lex}} v$

4. Refinement of Pairwise Involutive Divisions

Refinement of involutive divisions

Definition (refinement) Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be two distinct involutive divisions. We shall say that division $\mathcal{L}_2$ refines division $\mathcal{L}_1$ if the following relation holds

$$(\forall U \subset \mathcal{M})\ \ (\forall u \in U)\ \ [\,NM_{\mathcal{L}_2}(u,U) \subseteq NM_{\mathcal{L}_1}(u,U)\,]\ .$$

Corrolary If involutive division $\mathcal{L}_2$ refines division $\mathcal{L}_1$ and $U \subset \mathcal{M}$, then the corresponding minimal $\mathcal{L}_1-$basis $\bar{U}_1$ and $\mathcal{L}_2-$basis $\bar{U}_2$ satisfy

$$\bar{U}_2 \subseteq \bar{U}_1\ .$$

This means that either $\bar{U}_2 = \bar{U}_1$ or $\bar{U}_2$ is more compact then $\bar{U}_1$.

Given a nonempty monomial set $U \subset \mathcal{M}$, we shall denote by $\mathrm{GB}(U)$ the minimal basis (i.e., the reduced Gröbner basis) of $\langle U \rangle$.

Definition (ancestor) Given $U \subset \mathcal{M}$, $u \in U$ and a total monomial ordering $\sqsupset$ on $\mathcal{M}$, the element $v \in \mathrm{GB}(U)$ is said to be an ancestor of $u \in U$ w.r.t. $\sqsubset$ (denotation: $v = \mathrm{anc}(u,U)$) if

$$v := \max_{\sqsupset}\{\, w \in \mathrm{GB}(U) \ \big|\ w \mid u \,\}$$

Refinement of $\sqsupset$-division

Let $\sqsupset$ be a total monomial ordering compatible with multiplication, U a finite monomial set U and $u, v \in U$. Then we define another monomial ordering denoted by $\sqsupset_{\mathrm{GB}}$ and given by

$$u \sqsupset_{\mathrm{GB}} v \ \ \mathbf{if} \ \ \ \mathrm{anc}(u,U) \sqsupset \mathrm{anc}(v,U) \ \ \mathbf{or} \ \ \big(\mathrm{anc}(u,U) = \mathrm{anc}(v,U) \ \ \mathbf{and} \ \ u \sqsupset v\big).$$

This ordering generates pairwise involutive division, $\sqsupset_{\mathrm{GB}}$-division.

Theorem $\sqsupset_{\mathrm{GB}}$-division is Noetherian, continuous and constructive. It refines $\sqsupset$-division.

Examples:

- $\succ_{\mathrm{lexGB}}$-division refines Janet division ($\succ_{\mathrm{lex}}$-division)
- $\succ_{\mathrm{grlexGB}}$-division refines $\succ_{\mathrm{grlex}}$-division
- $\succ_{\mathrm{alexGB}}$-division refines $\succ_{\mathrm{alex}}$-division

5. Conclusions

- We suggest a method of refinement for involutive divisions which are pairwise generated by total monomial orderings. In particular, the suggested method allows to refine the Janet division.
- $\succ_{\mathrm{alex}}$-division, as a representative of this class, is heuristically better than Janet division. The last in its turn is heuristically better then other divisions generated by admissible orderings, i.e. $\succ_{\mathrm{grlex}}$-division.

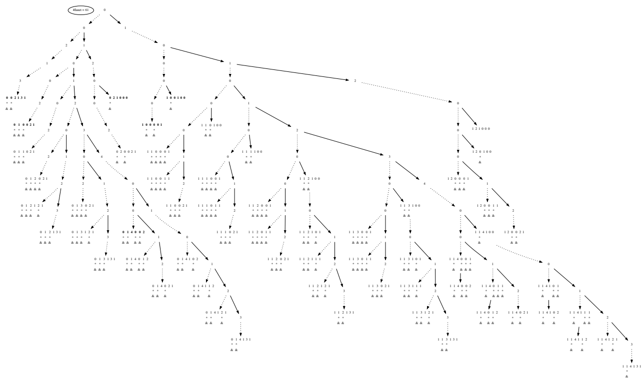

FIGURE 1. Janet tree #61 monoms

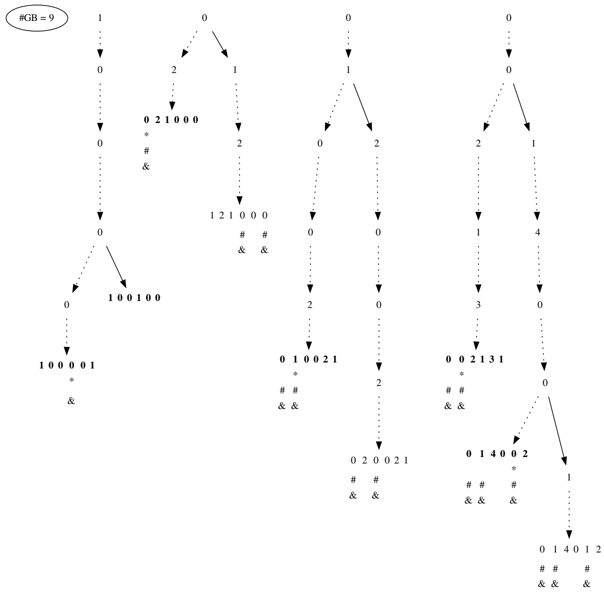

FIGURE 2. Janet forest #9 monoms

- Computational superiority of $\succ_{\mathrm{alex}}$-division over Janet division is expressed not only in a smaller number of nonmultplicative prolongations (number of the involutive normal forms evaluated) to be examined but also in a higher stability under permutation of the variables.

- The $\mathcal{L}_{\rm GB}-$division yields more compact involutive bases than the corresponding pairwise involutive division. Using the Janet forest, it allows you to quickly find the involutive divisor and define nonmulticative variables.
- Among $\mathcal{L}_{\rm GB}-$divisions the most compact bases generated by the antigraded orderings $\sqsupset$, such as $\succ_{{\rm alex}_{\rm GB}}$ $-$partition.

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