# Compact involutive division based on the Janet tree

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Gerdt (2001): Pair property of involutive division.
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Hashemi, Parnian (2018): D-Noether division.

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Research problem: efficient construction of most compact involutive bases.

# Implementation

Schwarz (1984): Riquier-Janet theory in Reduce.
Schwarz (1992): Linear differential Janet bases (DJB) in Reduce.
Zharkov, Blinkov (1993); G., Blinkov (1995): Pol. Pommaret bases (PPB) in Reduce.
Kredel (1996): PPB in MAS.
Nischke (1996): Polynomial JB (PJB) and PPB in C++ (PoSSoLib).
Berth (1999): Polynomial and differential involutive bases in Mathematica.
Cid (2000)-Robertz (2002-2011): PJB, DJB and difference JB in Maple.
Blinkov (2000-2007): PJB in Reduce, C++, GInv.
Yanovich (2001-2004): PJB in C, Singular.
Hausdorf, Seiler (2000-2002): DJB and DPB in MuPAD.
Chen, Gao (2002): Involutive extended characteristic sets in Maple.
Hemmecke (2002): Sliced division algorithm in Aldor.
Evans (2005): Noncommutative Involutive Bases in C.
Zhang, Li (2005): Janet bases for linear differential ideals in Maple.
Zinin (2007) - Blinkov (2011): Boolean Janet and Pommaret bases in C++, Reduce, Macaulay.
Langer-Hegermann (2010): Janet Bases for nonlinear and algebraically simple differential systems in Maple.
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# Involutive Monomial Division

An involutive division $\mathcal{L}$ is defined on $\mathcal{M} := \{ x_1^{i_1} \cdots x_n^{i_n} \mid i_j \in \mathbb{N}_{\geq 0} \}$
if for any finite $U \subset \mathcal{M}$ and $\forall u \in U$ defined a submonoid $\mathcal{L}(u, U)$ of $\mathcal{M}$ s.t.

1. $w \in \mathcal{L}(u, U),\ v|w \Longrightarrow v \in \mathcal{L}(u, U)$ (partition of variables)
2. $u, v \in U,\ u\mathcal{L}(u, U) \cap v\mathcal{L}(v, U) \neq \emptyset \Longrightarrow u \in v\mathcal{L}(v, U) \vee v \in u\mathcal{L}(u, U)$ ("unicity" of $\mathcal{L}$-divisor)
3. $v \in U,\ v \in u\mathcal{L}(u, U) \Longrightarrow \mathcal{L}(v, U) \subseteq \mathcal{L}(u, U)$ (transitivity)
4. $V \subseteq U \Longrightarrow \mathcal{L}(u, U) \subseteq \mathcal{L}(u, V)\ \forall u \in V$ (filter axiom)

Elements of $\mathcal{L}(u, U)$ are ($\mathcal{L}-$)multiplicative for $u$.

$$u \in U \quad \Rightarrow \quad \{x_1, \ldots, x_n\} = \underbrace{M_{\mathcal{L}}(u, U)}_{\text{multiplicative}} \biguplus \underbrace{NM_{\mathcal{L}}(u, U)}_{\text{nonmultiplicative}}$$

$\boxed{w \in u\mathcal{L}(u, U) \Longleftrightarrow u \mid_{\mathcal{L}} w}$ $u$ is involutive divisor ($\mathcal{L}-$divisor) of $w$

$\mathcal{C}_{\mathcal{L}}(u, U) := u\mathcal{L}(u, U)$ is involutive cone ($\mathcal{L}-$cone) generated by $u \in U$.

# Involutive bases

Given an ideal $\mathcal{I} \subset \mathcal{K}[x_1, \ldots, x_n]$, involutive division $\mathcal{L}$ and monomial order $\succ$, a finite subset $G \subset \mathcal{I}$ is called $(\mathcal{L})-$involutive basis of $\mathcal{I}$ if

$$(\forall f \in \mathcal{I})\,(\exists\, g \in G)\ \ [\,\mathrm{lm}(g) \mid_{\mathcal{L}} \mathrm{lm}(f)\,]$$

$$\Updownarrow \quad (\text{ for continuous } \mathcal{L}\ )$$

$$(\ \forall f \in G\ )\ (\ \forall x_i \in NM_{\mathcal{L}}(\mathrm{lm}(f), \mathrm{lm}(G))\ )\ \ [\ NF_{\mathcal{L}}(\underbrace{x_i \cdot f}, G) = 0\ ]$$

$$\uparrow$$

nonmultiplicative prolongation

$$\underbrace{\mathrm{lm}(g) \mid_{\mathcal{L}} \mathrm{lm}(f) \Longrightarrow \mathrm{lm}(g) \mid \mathrm{lm}(f)}$$

$$\Downarrow$$

An involutive basis is a Gröbner basis (GB), generally, redundant.

Similarly to a reduced GB a monic minimal involutive basis is unique.

# Monomial completion algorithm

**Input:** $U$, a finite set or list of monomials in $\mathcal{M}$; $\mathcal{L}$, an involutive division

**Output:** $\bar{U}$, a minimal $\mathcal{L}$-basis of $\langle U \rangle$

**choose** $u \in U$ without proper divisors in $U \setminus \{u\}$
$W := \{u\}; \;\; V := U \setminus \{u\} \cup \{\, u \cdot x \mid x \in NM_{\mathcal{L}}(u, W) \,\}$
**while** $V \neq \emptyset$ **do**
    **choose** $v \in V$ without proper divisors in $V \setminus \{v\}$
    $V := V \setminus \{v\}$
    **if** $v \notin C_{\mathcal{L}}(W)$ **then**
        $W := W \cup \{v\}; \;\; V := V \cup \{\, u \cdot x \mid u \in W, x \in NM_{\mathcal{L}}(u, W) \,\}$
    **fi**
**od**
**return** $\bar{U} := W$

# Computational efficiency of monomial completion

$\mathcal{L}$-size of $U$ is the total number of $\mathcal{L}$-nonmultiplicative variables for the elements in $U$

$$\mathcal{L}(U) := \sum_{u \in U} |NM_{\mathcal{L}}(u, U)|$$

**Proposition (Gerdt, Blinkov'11)**

Let $\mathcal{L}$ be noetherian and constructive, and let the algorithm completes $U$ to $\bar{U}$. Then $\bar{U}$ is produced from $U$ by running the while-loop exactly $\mathcal{L}(\bar{U}) + |U \setminus \bar{U}| - 1$ times provided repeated nonmultiplicative prolongations are avoided.

Thus, $\mathcal{L}(\bar{U})$ measures computational efficiency of $\mathcal{L}$.

**Remark**

$\mathcal{L}_1(\bar{U}_1) < \mathcal{L}_2(\bar{U}_2)$ strongly correlates with $|\bar{U}_1| < |\bar{U}_2|$.

# Construction of involutive divisions

All known involutive divisions satisfying the above Axioms 1-3 are pairwise ( Gerdt'01 ), i.e. for any finite set $U \subset \mathcal{M}$ with cardinality $|U| \geq 2$ the set of its $\mathcal{L}$-nonmultiplicative variables is given by

$$( \forall u \in U ) \;\; [ \, NM_{\mathcal{L}}(u, U) = \bigcup_{v \in U \setminus \{u\}} NM_{\mathcal{L}}(u, \{u, v\}) \, ]$$

The pair property provides a regular procedure for construction of a pairwise involutive division called $\sqsupset$-division ( Gerdt,Blinkov'11 ) if it is generated by a total monomial order $\sqsupset$ under the fixed permutation $\sigma$ on the variables

$$NM_{\sqsupset}(u, \{u, v\}) := \begin{cases} \text{if } u \sqsupset v \text{ or } (u \sqsubset v \wedge v \mid u) \text{ then } \emptyset \\ \text{else } \{x_{\sigma(i)}\}, \; i = \min\{j \mid \deg_{\sigma(j)}(u) < \deg_{\sigma(j)}(v)\} \end{cases}$$

## Remark

There are $n!$ distinct $\sqsubset$-divisions where $n$ is a number variables.

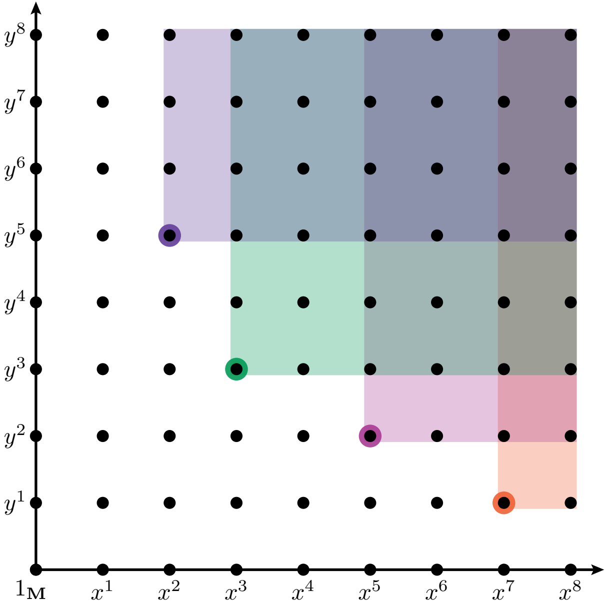
# Properties of $\sqsupset$-divisions

If $\sqsupset$ is admissible or the negation of admissible, then $\sqsupset$-division is continuous, constructive and Noetherian ( Semenov'06, Semenov,Zyuzikov'07-08, Gerdt,Blinkov'11 )

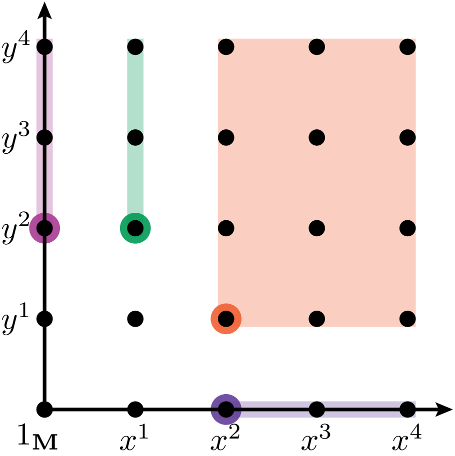
Some particular divisions:

- Janet division generated by the lexicographic order $\succ_{\rm lex}$
- $\succ_{\rm grlex}$-division generated by the graded lexicographic order: $u \succ_{\rm grlex} v \Longleftrightarrow \deg(u) > deg(v) \ \vee \ deg(u) = deg(v) \ \wedge \ u \succ_{\rm lex} v$
- $\succ_{\rm alex}$-division generated by the antigraded lexicographic order: $u \succ_{\rm alex} v \Longleftrightarrow \deg(u) < deg(v) \ \vee \ deg(u) = deg(v) \ \wedge \ u \succ_{\rm lex} v$

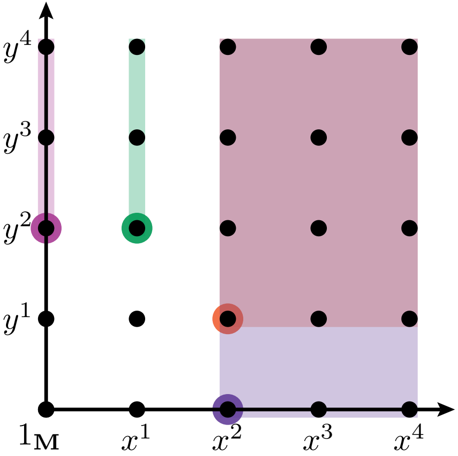
# Multivariate monomial division: overlapped cones



# Disjoint and embedded involutive cones




$\mathcal{C}_{\text{Janet}}\{x^2y, x^2, xy^2, y^2\}$




$\mathcal{C}_{\succ_{\text{alex}}}\{x^2y, x^2, xy^2, y^2\}$

# Refinement of involutive divisions

## Definition (refinement)

Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be two distinct involutive divisions. We shall say that division $\mathcal{L}_2$ refines division $\mathcal{L}_1$ if the following relation holds

$$(\forall U \subset \mathcal{M})\ (\forall u \in U)\ [\, NM_{\mathcal{L}_2}(u, U) \subseteq NM_{\mathcal{L}_1}(u, U)\,]\ .$$

## Corrolary

If involutive division $\mathcal{L}_2$ refines division $\mathcal{L}_1$ and $U \subset \mathcal{M}$, then the corresponding minimal $\mathcal{L}_1-$basis $\bar{U}_1$ and $\mathcal{L}_2-$basis $\bar{U}_2$ satisfy

$$\bar{U}_2 \subseteq \bar{U}_1\ .$$

This means that either $\bar{U}_2 = \bar{U}_1$ or $\bar{U}_2$ is more compact then $\bar{U}_1$.

# Ancestors

Given a nonempty monomial set $U \subset \mathcal{M}$, we shall denote by $\mathrm{GB}(U)$ the minimal basis (i.e., the reduced Gröbner basis) of $\langle U \rangle$.

## Definition (ancestor)

Given $U \subset \mathcal{M}$, $u \in U$ and a total monomial ordering $\sqsupset$ on $\mathcal{M}$, the element $v \in \mathrm{GB}(U)$ is said to be an ancestor of $u \in U$ w.r.t. $\sqsubset$ (denotation: $v = \mathrm{anc}(u, U)$) if

$$v := \max_{\sqsupset}\{\, w \in \mathrm{GB}(U) \;\big|\; w \mid u \,\}$$

# Refinement of $\sqsupset$-division

## Definition

Let $\sqsupset$ be a total monomial ordering compatible with multiplication, $U$ a finite monomial set $U$ and $u, v \in U$. Then we define another monomial ordering denoted by $\sqsupset_{\mathrm{GB}}$ and given by

$$u \sqsupset_{\mathrm{GB}} v \ \ \textbf{if} \ \ \mathrm{anc}(u,U) \sqsupset \mathrm{anc}(v,U) \ \ \textbf{or} \ \ \big(\mathrm{anc}(u,U) = \mathrm{anc}(v,U) \ \ \textbf{and} \ \ u \sqsupset v\big).$$

This ordering generates pairwise involutive division, $\sqsupset_{\mathrm{GB}}$-division.

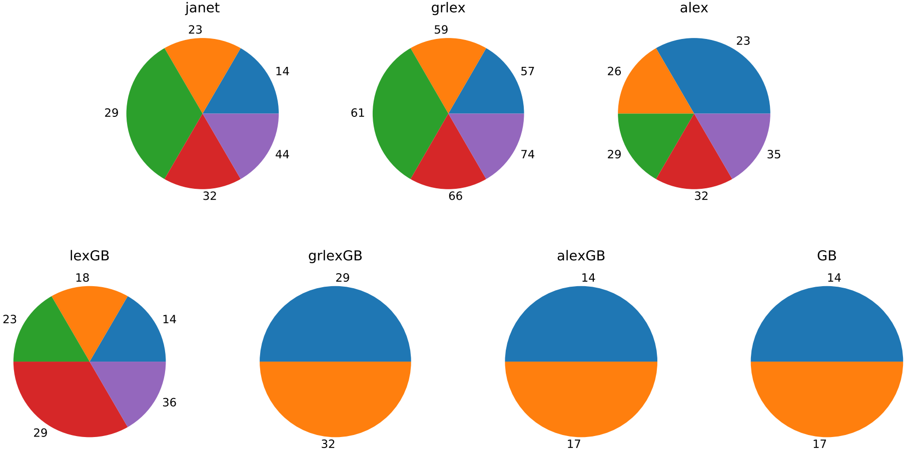
## Theorem

$\sqsupset_{\mathrm{GB}}$-division is Noetherian, continuous and constructive. It refines $\sqsupset$-division.

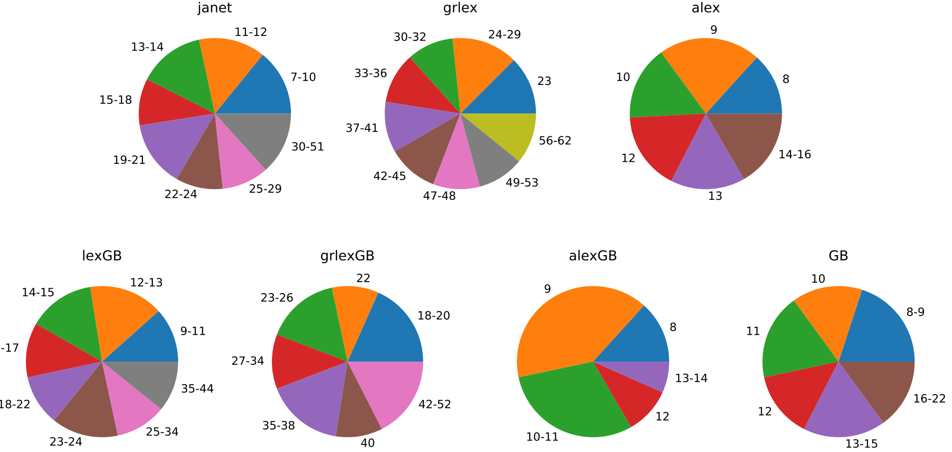
## Examples

- $\succ_{\mathrm{lexGB}}$-division refines Janet division ( $\succ_{\mathrm{lex}}$-division )
- $\succ_{\mathrm{grlexGB}}$-division refines $\succ_{\mathrm{grlex}}$-division
- $\succ_{\mathrm{alexGB}}$-division refines $\succ_{\mathrm{alex}}$-division

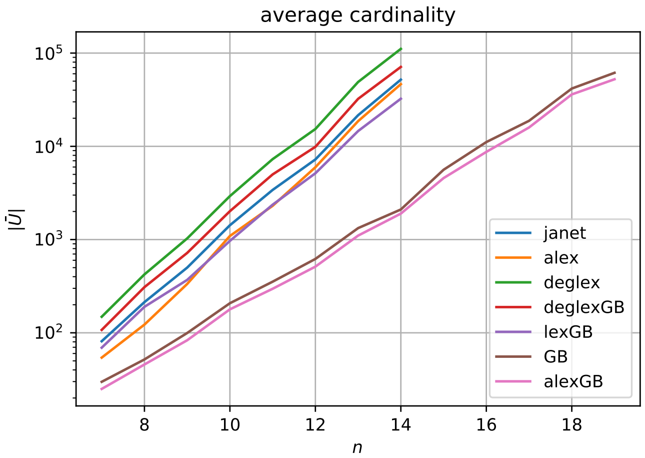
# Completion of $\{\, x_1^{10}, x_1^7 x_2, x_1^4 x_2^2, x_1 x_2 x_3^3, x_2^4 \,\}$



# Completion of $\{ x_1^2 x_2^2 x_5, x_2^2 x_3 x_5, x_2 x_4, x_3^2, x_3 x_4 x_5^2 \}$

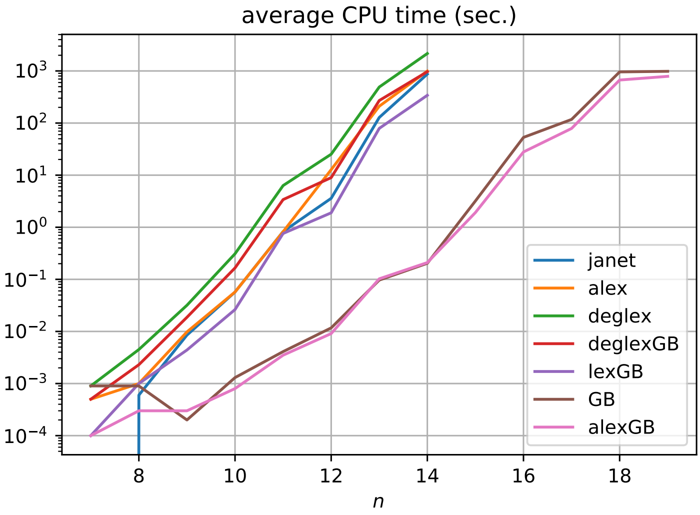


# Cardinality of Involutive Monomial Bases




For every $n$ we generated a random set of 100 monomials in $n$ variables of the minimal degree 2 and the maximal one $3/2n$.

# Computational Efficiency

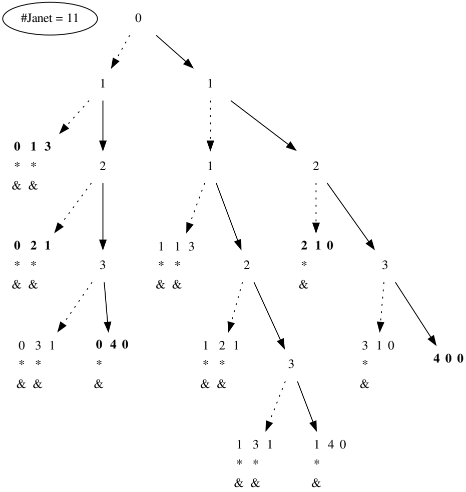


| division | # on average | CPU time |
|---|---|---|
| | $n = 7$ | |
| GB | 30 | 0.0000 |
| alex | 54 | 0.0005 |
| alexGB | 25 | 0.0001 |
| deglex | 149 | 0.0009 |
| deglexGB | 107 | 0.0005 |
| janet | 81 | 0.0000 |
| lexGB | 69 | 0.0001 |
| | $n = 8$ | |
| GB | 52 | 0.0000 |
| alex | 123 | 0.0010 |
| alexGB | 46 | 0.0003 |
| deglex | 423 | 0.0045 |
| deglexGB | 309 | 0.0023 |
| janet | 213 | 0.0006 |
| lexGB | 190 | 0.0010 |

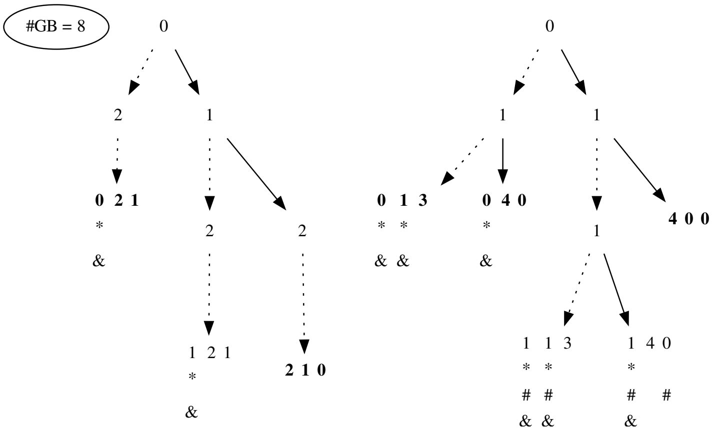
| division | # on average | CPU time |
|---|---|---|
| | $n = 9$ | |
| GB | 100 | 0.0002 |
| alex | 334 | 0.0098 |
| alexGB | 83 | 0.0003 |
| deglex | 1029 | 0.0318 |
| deglexGB | 719 | 0.0187 |
| janet | 500 | 0.0085 |
| lexGB | 369 | 0.0044 |
| | $n = 10$ | |
| GB | 208 | 0.0013 |
| alex | 1093 | 0.0572 |
| alexGB | 179 | 0.0008 |
| deglex | 2925 | 0.3099 |
| deglexGB | 2016 | 0.1656 |
| janet | 1424 | 0.0571 |
| lexGB | 968 | 0.0263 |

| division | # on average | CPU time |
|---|---|---|
| $n = 11$ | | |
| GB | 354 | 0.0041 |
| alex | 2302 | 0.8206 |
| alexGB | 297 | 0.0035 |
| deglex | 7289 | 6.3102 |
| deglexGB | 4997 | 3.3944 |
| janet | 3407 | 0.8256 |
| lexGB | 2367 | 0.7598 |
| $n = 12$ | | |
| GB | 620 | 0.0117 |
| alex | 5962 | 12.7423 |
| alexGB | 513 | 0.0091 |
| deglex | 15337 | 25.2190 |
| deglexGB | 9877 | 8.9664 |
| janet | 7231 | 3.5894 |
| lexGB | 5141 | 1.8921 |

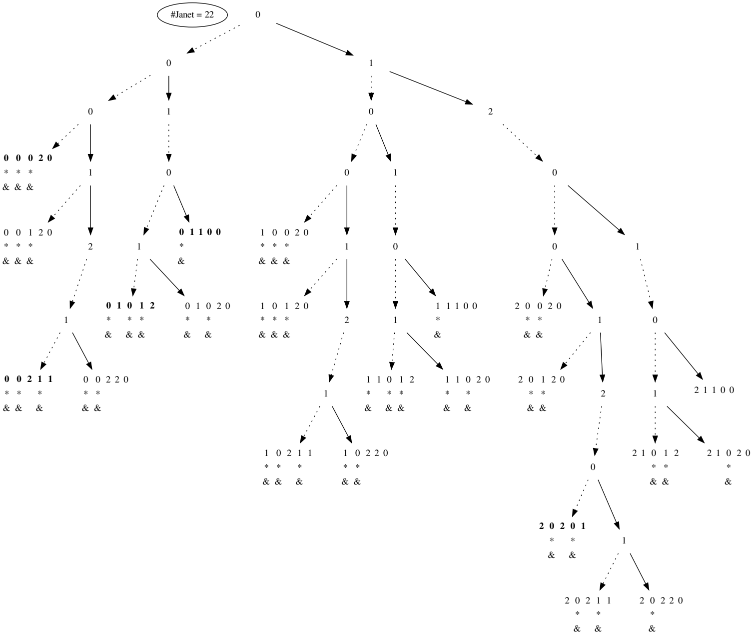
| division | # on average | CPU time |
|---|---|---|
| $n = 13$ | | |
| GB | 1327 | 0.0968 |
| alex | 18625 | 210.3615 |
| alexGB | 1103 | 0.1024 |
| deglex | 48969 | 491.2970 |
| deglexGB | 32298 | 270.6196 |
| janet | 21564 | 127.9710 |
| lexGB | 14581 | 79.0166 |
| $n = 14$ | | |
| GB | 2102 | 0.2033 |
| alex | 46437 | 993.7210 |
| alexGB | 1899 | 0.2098 |
| deglex | 110923 | 2162.4816 |
| deglexGB | 71060 | 967.3826 |
| janet | 51913 | 870.1882 |
| lexGB | 32400 | 340.9174 |

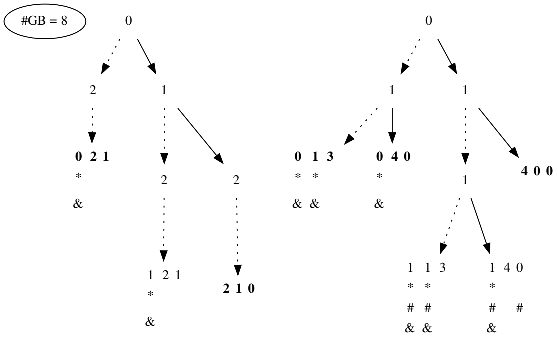



Janet tree #8 monoms

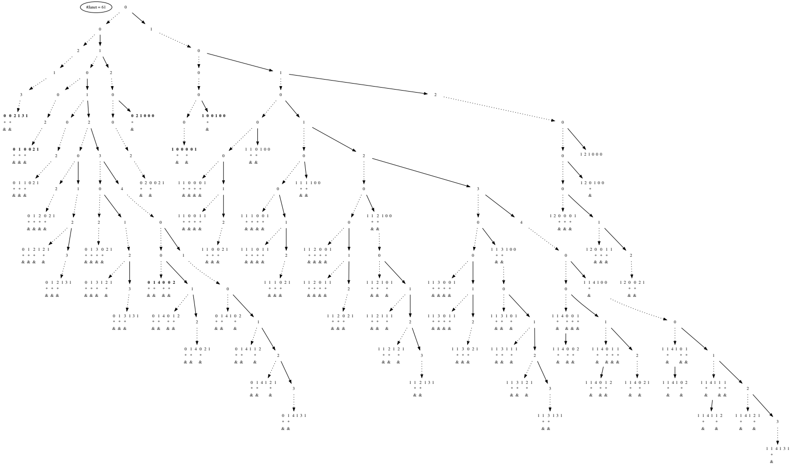



Janet forest #11 monoms

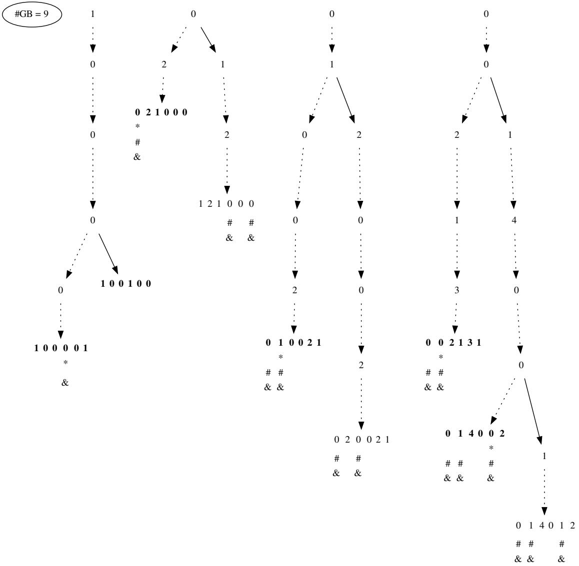

#Janet = 22
0 0 0 2 0
0 0 1 2 0
0 0 2 1 1
0 0 2 2 0
0 1 0 1 2
0 1 0 2 0
0 1 1 0 0
1 0 0 2 0
1 0 1 2 0
1 0 2 1 1
1 0 2 2 0
1 1 0 1 2
1 1 0 2 0
1 1 1 0 0
2 0 0 2 0
2 0 1 2 0
2 0 2 0 1
2 0 2 1 1
2 0 2 2 0
2 1 0 1 2
2 1 0 2 0
2 1 1 0 0




Janet forest #8 monoms



# Janet tree #61 monoms




# Janet forest #9 monoms

# Conclusions

- We suggest a method of refinement for involutive divisions which are pairwise generated by total monomial orderings. In particular, the suggested method allows to refine the Janet division.
- $\succ_{\mathrm{alex}}$-division, as a representative of this class, is heuristically better than Janet division. The last in its turn is heuristically better then other divisions generated by admissible orderings, i.e. $\succ_{\mathrm{grlex}}$-division.
- Computational superiority of $\succ_{\mathrm{alex}}$-division over Janet division is expressed not only in a smaller number of nonmultplicative prolongations (number of the involutive normal forms evaluated) to be examined but also in a higher stability under permutation of the variables.
- The $\mathcal{L}_{\mathrm{GB}}-$division yields more compact involutive bases than the corresponding pairwise involutive division. Using the Janet forest, it allows you to quickly find the involutive divisor and define nonmulticative variables.
- Among $\mathcal{L}_{\mathrm{GB}}-$divisions the most compact bases generated by the antigraded orderings $\sqsupset$, such as $\succ_{\mathrm{alex}_{\mathrm{GB}}}$ $-$partition.