Using Tropical Algebra to Solve a Minimax Optimization Problem in Project Scheduling

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Abstract. We apply tropical optimization techniques to derive a complete analytical solution to a time-constrained project scheduling problem.

1. Time-Constrained Project Scheduling Problem

Consider a project that involves n activities (jobs, tasks, operations) to be performed in parallel, subject to a set of temporal constraints, including the release time and deadline constraints, as well as start-start and start-finish relationships. The scheduling problem is to find an optimal schedule that minimizes the maximum deviation of the start time of all activities.

For each activity $i = 1, \ldots, n$, we denote the start times by x_i and the finish time by y_i. Given release times g_i that specify the earliest allowed time for activities to start and deadlines f_i that define the latest time for activities to finish, the release time and deadline constraints take the form of the inequalities

$$g_i \leq x_i, \quad y_i \leq f_i, \qquad i = 1, \ldots, n.$$

The start-start constraints indicate the minimum allowed time lag between the start of one activity and the start of another. Given parameters b_{ij}, the start-start constraints are represented as

$$\max_{1 \leq j \leq n} (b_{ij} + x_j) \leq x_i, \qquad i = 1, \ldots, n.$$

The start-finish constraints determine the minimum allowed time lag between the start of one activity and the finish of another. Given parameters a_{ij}, the start-finish constraints are represented as

$$\max_{1 \leq j \leq n} (c_{ij} + x_j) \leq y_i, \qquad i = 1, \ldots, n.$$

Considering the deadlines $y_i \leq f_i$, these constraints can be replaced by

$$\max_{1 \leq j \leq n} (c_{ij} + x_j) \leq f_i, \qquad i = 1, \ldots, n.$$

If some parameters g_i, b_{ij} and c_{ij} are not defined, we set them equal to $-\infty$.

The maximum deviation of the start time of activities is defined as the interval between the earliest and the latest start times of activities and is given by

$$\max_{1\leq i\leq n} x_i - \min_{1\leq i\leq n} x_i = \max_{1\leq i\leq n} x_i + \max_{1\leq i\leq n}(-x_i).$$

The scheduling problem with the parameters g_i, h_i, b_{ij} and c_{ij} given for all $i,j=1,\dots,n$, is to find start times x_i that solve the minimax problem

$$\begin{aligned} \min_{x_1,\dots,x_n} \quad & \max_{1\leq i\leq n} x_i + \max_{1\leq i\leq n}(-x_i), \\ \text{s.t.} \quad & \max_{1\leq j\leq n}(b_{ij}+x_j) \leq x_i, \quad \max_{1\leq j\leq n}(c_{ij}+x_j) \leq f_i, \\ & g_i \leq x_i, \quad i=1,\dots,n. \end{aligned} \tag{1}$$

We observe that problem (1) can be represented as a linear program, and then a numerical solution of the problem can be obtained using one of the computational algorithms available in linear programming. To derive a complete analytical solution to the problem, we formulate it in terms of tropical algebra and apply methods and results of tropical optimization.

2. Elements of Tropical Algebra

Let $\mathbb{X}$ be a set that is equipped with two distinct elements: zero $\mathbb{0}$ and identity $\mathbb{1}$, and closed under associative and commutative binary operations: addition $\oplus$ and and multiplication $\otimes$. Addition is idempotent which means that $x\oplus x=x$ for all $x\in\mathbb{X}$. Multiplication distributes over addition and is invertible which provides each nonzero $x\in\mathbb{X}$ with an inverse $x^{-1}\in\mathbb{X}$ such that $x\otimes x^{-1}=\mathbb{1}$. Finally, the set $\mathbb{X}$ is assumed to be totally ordered by an order relation consistent with that induced by idempotent addition using the rule: $x\leq y$ if and only if $x\oplus y=y$.

The algebraic system $(\mathbb{X},\oplus,\otimes,\mathbb{0},\mathbb{1})$ is usually referred to as the idempotent semifield. In what follows, the multiplication sign is omitted to save writing.

As an example, one can consider the real semifield $(\mathbb{R}\cup\{-\infty\},\max,+,-\infty,0)$, which is usually called the max-plus algebra. In this semifield, the operations are defined as $\oplus=\max$ and $\otimes=+$, and the neutral elements as $\mathbb{0}=-\infty$ and $\mathbb{1}=0$. For any $x\in\mathbb{R}$, the multiplicative inverse is equal to the opposite number $-x$ in the standard representation. The power x^y coincides with the usual product $x\times y$. The order relation $\leq$ corresponds to the natural linear order on $\mathbb{R}$.

The scalar operations $\oplus$ and $\otimes$ are extended to vectors and matrices over $\mathbb{X}$ in the usual way. All vectors are assumed column vectors unless transposed. A matrix (vector) with all entries equal to $\mathbb{0}$ is the zero matrix (vector). A vector that has no zero elements is called regular. Any matrix without zero columns is called column-regular. A vector with all elements equal to $\mathbb{1}$ is denoted by $\mathbf{1}=(\mathbb{1},\dots,\mathbb{1})^T$. Multiplicative conjugate transposition of a nonzero vector $\boldsymbol{x}=(x_i)$ yields the row vector $\boldsymbol{x}^-=(x_i^-)$, where $x_i^-=x_i^{-1}$ if $x_i\neq\mathbb{0}$ and $x_i^-=\mathbb{0}$ otherwise.

A square matrix that has all diagonal elements equal to $\mathbb{1}$ and off-diagonal to $\mathbb{0}$, is the identity matrix denoted by $\boldsymbol{I}$. The power notation is defined as follows: $\boldsymbol{A}^0 = \boldsymbol{I}$, $\boldsymbol{A}^p = \boldsymbol{A}\boldsymbol{A}^{p-1}$ for any square matrix $\boldsymbol{A}$ and integer $p > 0$.

The trace of a square matrix $\boldsymbol{A} = (a_{ij})$ of order n is given by

$$\operatorname{tr}(\boldsymbol{A}) = a_{11} \oplus \cdots \oplus a_{nn} = \bigoplus_{i=1}^{n} a_{ii}.$$

A tropical analogue of the matrix determinant is defined as

$$\operatorname{Tr}(\boldsymbol{A}) = \operatorname{tr} \boldsymbol{A} \oplus \cdots \oplus \operatorname{tr} \boldsymbol{A}^n = \bigoplus_{m=1}^{n} \operatorname{tr} \boldsymbol{A}^m.$$

If the condition $\operatorname{Tr}(\boldsymbol{A}) \leq \mathbb{1}$ holds, the Kleene star matrix is calculated as

$$\boldsymbol{A}^* = \boldsymbol{I} \oplus \boldsymbol{A} \oplus \cdots \oplus \boldsymbol{A}^{n-1} = \bigoplus_{m=0}^{n-1} \boldsymbol{A}^m.$$

For any matrix $\boldsymbol{A} = (a_{ij})$ and vector $\boldsymbol{x} = (x_i)$, tropical norms are given by

$$\|\boldsymbol{A}\| = \boldsymbol{1}^T \boldsymbol{A} \boldsymbol{1} = \bigoplus_{i,j} a_{ij}, \qquad \|\boldsymbol{x}\| = \boldsymbol{1}^T \boldsymbol{x} = \boldsymbol{x}^T \boldsymbol{1} = \bigoplus_{i} x_i,$$

which coincide in max-plus algebra with the maximum entries of $\boldsymbol{A}$ and $\boldsymbol{x}$.

3. Algebraic Solution of Scheduling Problem

We now present a complete analytical solution of the scheduling problem of interest, obtained in the framework of tropical algebra [1]. Other related examples of applications of tropical optimization can be found, e.g., in [2, 3, 4, 5].

Consider the minimax optimization problem at (1) and represent the problem in scalar form in terms of max-plus algebra to write

$$\begin{aligned} \min_{x_1,\ldots,x_n} \quad & \bigoplus_{1\leq i\leq n} x_i \bigoplus_{1\leq j\leq n} x_j^{-1}, \\ \text{s.t.} \quad & \bigoplus_{1\leq j\leq n} b_{ij} x_j \leq x_i, \quad \bigoplus_{1\leq j\leq n} c_{ij} x_j \leq f_i, \\ & g_i \leq x_i, \quad i = 1,\ldots,n. \end{aligned} \tag{2}$$

With the following matrix-vector notation:

$$\boldsymbol{B} = (b_{ij}), \qquad \boldsymbol{C} = (c_{ij}), \qquad \boldsymbol{x} = (x_i), \qquad \boldsymbol{g} = (g_i), \qquad \boldsymbol{f} = (f_i),$$

problem (2) takes the vector form

$$\begin{aligned} \min_{\boldsymbol{x}} \quad & \boldsymbol{x}^{-} \boldsymbol{1} \boldsymbol{1}^T \boldsymbol{x}, \\ \text{s.t.} \quad & \boldsymbol{B}\boldsymbol{x} \leq \boldsymbol{x}, \quad \boldsymbol{C}\boldsymbol{x} \leq \boldsymbol{f}, \\ & \boldsymbol{g} \leq \boldsymbol{x}. \end{aligned} \tag{3}$$

Application of tropical optimization techniques developed in [2, 3, 4, 5] leads to the next result, which offers a complete analytical solution to problem (3).

Lemma 1. *Let* $\boldsymbol{B}$ *be a matrix with* $\mathrm{Tr}(\boldsymbol{B}) \leq \mathbb{1}$ *and* $\boldsymbol{C}$ *a column-regular matrix. Let* $\boldsymbol{g}$ *be a vector and* $\boldsymbol{f}$ *a regular vector such that* $\boldsymbol{f}^{-}\boldsymbol{C}\boldsymbol{B}^{*}\boldsymbol{g} \leq \mathbb{1}$.

Then, the minimum of the objective function in problem (3) *is equal to*

$$\theta = \|\boldsymbol{B}^{*}\| \oplus \bigoplus_{0 \leq i+j \leq n-2} \|\boldsymbol{f}^{-}\boldsymbol{C}\boldsymbol{B}^{i}\| \|\boldsymbol{B}^{j}\boldsymbol{g}\|,$$

and all regular solutions are given in parametric form by

$$\boldsymbol{x} = \boldsymbol{G}\boldsymbol{u}, \qquad \boldsymbol{g} \leq \boldsymbol{u} \leq (\boldsymbol{f}^{-}\boldsymbol{C}\boldsymbol{G})^{-},$$

where $\boldsymbol{u}$ *is a vector of parameters and*

$$\boldsymbol{G} = \boldsymbol{B}^{*} \oplus \bigoplus_{0 \leq i+j \leq n-2} \theta^{-1}\boldsymbol{B}^{i}\boldsymbol{1}\boldsymbol{1}^{T}\boldsymbol{B}^{j}.$$

This work was supported in part by the Russian Foundation for Basic Research grant number 20-010-00145.

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