Using Tropical Algebra to Solve a Minimax Optimization Problem in Project Scheduling

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Outline

Project Scheduling: Constraints and Objective

Temporal Constraints

- Consider a project that involves n activities (tasks, operations) to be performed in parallel, subject to a set of temporal constraints
- The release times define the earliest time for activities to start
- The deadlines determine the latest time for activities to finish
- The start-start constraints specify the minimum allowed time lag between the start of one activity and the start of another
- The start-finish constraints indicate the minimum allowed time lag between the start of one activity and the finish of another

Scheduling Objective

- The maximum deviation of start times is the interval between the earliest and the latest start times of activities
- The scheduling objective of interest is to minimize the maximum deviation of start times of activities in the project
- Such an objective arises when the scheduling problem is to provide, as much as possible, simultaneous start of all activities

Notation and Formal Representation

Notation

- For each activity $i = 1, \ldots, n$, the following notation is used:

x_i, *the unknown start time*;

y_i, *the unknown finish time*;

g_i, *the release time*;

f_i, *the deadline*;

b_{ij}, *the minimum possible time lag between the start of activity* $j = 1, \ldots, n$ *and the start of* i *(*$b_{ij} = -\infty$ *if unspecified)*;

c_{ij}, *the minimum possible time lag between the start of activity* $j = 1, \ldots, n$ *and the finish of* i *(*$c_{ij} = -\infty$ *if unspecified)*

Constraints

- The release time and deadline constraints:

$$g_i \leq x_i, \quad y_i \leq f_i, \qquad i = 1, \ldots, n$$

- The start-start constraints:

$$\max_{1 \leq j \leq n} (b_{ij} + x_j) \leq x_i, \qquad i = 1, \ldots, n$$

- The start-finish constraints:

$$\max_{1 \leq j \leq n} (c_{ij} + x_j) \leq y_i, \qquad i = 1, \ldots, n$$

- By combining with $y_i \leq f_i$, the start-finish constraints turn into

$$\max_{1 \leq j \leq n} (c_{ij} + x_j) \leq f_i, \qquad i = 1, \ldots, n$$

Objective Function and Optimization Problem

- The maximum deviation of the start time of activities:

$$\max_{1\leq i\leq n} x_i - \min_{1\leq i\leq n} x_i = \max_{1\leq i\leq n} x_i + \max_{1\leq i\leq n}(-x_i)$$

- Given parameters b_{ij}, c_{ij}, g_i and f_i for $i,j=1,\ldots,n$, the aim of optimal scheduling is to find the start times x_i to solve the problem

$$\begin{aligned} \min_{x_1,\ldots,x_n} \quad & \max_{1\leq i\leq n} x_i + \max_{1\leq i\leq n}(-x_i), \\ \text{s.t.} \quad & \max_{1\leq j\leq n}(b_{ij}+x_j) \leq x_i, \quad \max_{1\leq j\leq n}(c_{ij}+x_j) \leq f_i, \\ & g_i \leq x_i, \quad i=1,\ldots,n \end{aligned}$$

- The problem can be readily represented as a linear program and then solved using computational algorithms of linear programming

Tropical Optimization: Introduction

- Tropical (idempotent) mathematics deals with the theory and applications of algebraic systems with idempotent operations
- A binary operation is idempotent, if applied to operands with the same value, it gives this value as output (example: $\max(x, x) = x$)
- Tropical optimization is concerned with problems that are formulated and solved in the tropical mathematics setting
- The tropical optimization problems find applications in many areas to provide new solutions to various old and novel problems in
  - *project scheduling, location analysis, transportation networks,*
  - *decision making, discrete event systems and other fields*

Max-Plus Algebra

- Max-plus algebra is the set of reals $\mathbb{R}$ with $-\infty$ adjoined, which is closed under two operations: addition $\oplus$ and multiplication $\otimes$
- Addition $\oplus$ has the neutral element $\mathbb{0} = -\infty$, and it is defined as

$$x \oplus y = \max(x, y)$$

- Addition possesses the idempotent property given by

$$x \oplus x = \max(x, x) = x$$

- Multiplication $\otimes$ has the neutral element $\mathbb{1} = 0$ and is defined as

$$x \otimes y = x + y$$

Idempotent Semifield

- Idempotent semifield: the algebraic system $\langle \mathbb{X}, \oplus, \otimes, \mathbb{0}, \mathbb{1} \rangle$
- The binary operations $\oplus$ and $\otimes$ are associative and commutative
- Addition $\oplus$ has zero $\mathbb{0}$ and is idempotent: $x \oplus x = x$ for all $x \in \mathbb{X}$
- Multiplication $\otimes$ has identity $\mathbb{1}$ and is distributive over addition
- Multiplication $\otimes$ is invertible: for each nonzero $x \in \mathbb{X}$, there exists an inverse $x^{-1} \in \mathbb{X}$ such that $x \otimes x^{-1} = \mathbb{1}$
  - *In max-plus algebra, the inverse x^{-1} corresponds to $-x$*
- Algebraic completeness: the equation $x^r = a$ is solvable for any $a \in \mathbb{X}$ and rational r (there exist powers with rational exponents)
  - *In max-plus algebra, the power x^y corresponds to xy*
- Inequality for geometric and arithmetic means: $(xy)^{1/2} \leq x \oplus y$
- Notational convention: the multiplication sign $\otimes$ is omitted

Vectors over Idempotent Semifield

- Addition of two column vectors $\boldsymbol{a} = (a_j)$ and $\boldsymbol{b} = (b_j)$ and multiplication of the vector $\boldsymbol{a}$ by a scalar x are defined as

$$\{\boldsymbol{a} \oplus \boldsymbol{b}\}_j = a_j \oplus b_j \qquad (= \max(a_j, b_j) \text{ in max-plus algebra}),$$
$$\{x\boldsymbol{a}\}_j = xa_j \qquad (= x + a_j \text{ in max-plus algebra})$$

- A vector without zero entries is regular (*finite* in max-plus algebra)
- The vector of $\mathbb{1}$'s is denoted by $\mathbf{1} = (\mathbb{1}, \ldots, \mathbb{1})^T$ $(= (0, \ldots, 0)^T)$
- Multiplicative conjugate transpose of a vector $\boldsymbol{a} = (a_j)$ is the row vector $\boldsymbol{a}^- = (a_j^-)$ with $a_j^- = a_j^{-1}$ if $a_j \neq \mathbb{0}$, and $a_j^- = \mathbb{0}$ otherwise
- Vector norm is calculated for a column vector $\boldsymbol{a} = (a_j)$ as

$$\|\boldsymbol{a}\| = \mathbf{1}^T\boldsymbol{a} = \boldsymbol{a}^T\mathbf{1} = \bigoplus_j a_j \qquad \left(= \max_j a_j\right)$$

Matrices over Idempotent Semifield

- Matrix addition and multiplication of matrices $\boldsymbol{A} = (a_{ij})$, $\boldsymbol{B} = (b_{ij})$ and $\boldsymbol{C} = (c_{ij})$, and scalar multiplication of $\boldsymbol{A}$ by x are given by

$$\{\boldsymbol{A} \oplus \boldsymbol{B}\}_{ij} = a_{ij} \oplus b_{ij} \qquad (= \max(a_{ij}, b_{ij})),$$

$$\{\boldsymbol{A}\boldsymbol{C}\}_{ij} = \bigoplus_{k} a_{ik} c_{kj} \qquad \left(= \max_{k}(a_{ik} + c_{kj})\right),$$

$$\{x\boldsymbol{A}\}_{ij} = x a_{ij} \qquad (= x + a_{ij})$$

- A tropical analogue of norm is calculated for a matrix $\boldsymbol{A} = (a_{ij})$ as

$$\|\boldsymbol{A}\| = \mathbf{1}^{T}\boldsymbol{A}\mathbf{1} = \bigoplus_{i,j} a_{ij} \qquad \left(= \max_{i,j} a_{ij}\right)$$

- A matrix without zero columns is called column-regular

Square Matrices

- The identity matrix $\boldsymbol{I}$ has $\mathbb{1}$'s on the diagonal and $\mathbb{0}$'s elsewhere (0's on the diagonal and $-\infty$'s elsewhere in max-plus algebra)
- Positive integer powers of a square matrix $\boldsymbol{A}$ are defined as

$$\boldsymbol{A}^0 = \boldsymbol{I}, \qquad \boldsymbol{A}^p = \boldsymbol{A}\boldsymbol{A}^{p-1}, \qquad p \geq 1$$

- The trace of a square matrix $\boldsymbol{A} = (a_{ij})$ of order n is given by

$$\operatorname{tr} \boldsymbol{A} = a_{11} \oplus \cdots \oplus a_{nn} = \bigoplus_{i=1}^{n} a_{ii} \qquad \left(= \max_{1 \leq i \leq n} a_{ii} \right)$$

- The trace possesses the usual properties:

$$\operatorname{tr}(\boldsymbol{A} \oplus \boldsymbol{B}) = \operatorname{tr} \boldsymbol{A} \oplus \operatorname{tr} \boldsymbol{B}, \qquad \operatorname{tr}(\boldsymbol{A}\boldsymbol{B}) = \operatorname{tr}(\boldsymbol{B}\boldsymbol{A}), \qquad \operatorname{tr}(x\boldsymbol{A}) = x \operatorname{tr} \boldsymbol{A}$$

Matrix Determinant, Kleene Star and Spectral Radius

- An idempotent analogue of matrix determinant is calculated as

$$\mathrm{Tr}(\boldsymbol{A}) = \mathrm{tr}\,\boldsymbol{A} \oplus \cdots \oplus \mathrm{tr}\,\boldsymbol{A}^n = \bigoplus_{m=1}^{n} \mathrm{tr}\,\boldsymbol{A}^m$$

- Provided that $\mathrm{Tr}(\boldsymbol{A}) \leq \mathbb{1}$, the Kleene star operator is given by

$$\boldsymbol{A}^* = \boldsymbol{I} \oplus \boldsymbol{A} \oplus \cdots \oplus \boldsymbol{A}^{n-1} = \bigoplus_{m=0}^{n-1} \boldsymbol{A}^m$$

- The spectral radius of the matrix $\boldsymbol{A}$ is calculated as

$$\lambda = \mathrm{tr}\,\boldsymbol{A} \oplus \cdots \oplus \mathrm{tr}^{1/n}(\boldsymbol{A}^n) = \bigoplus_{m=1}^{n} \mathrm{tr}^{1/m}(\boldsymbol{A}^m)$$

Representation of Scheduling Problem

- Consider the scheduling problem in conventional scalar form

$$
\begin{aligned}
\min_{x_1,\ldots,x_n} \quad & \max_{1\leq i\leq n} x_i + \max_{1\leq i\leq n}(-x_i), \\
\text{s.t.} \quad & \max_{1\leq j\leq n}(b_{ij}+x_j) \leq x_i, \quad \max_{1\leq j\leq n}(c_{ij}+x_j) \leq f_i, \\
& g_i \leq x_i, \quad i = 1,\ldots,n
\end{aligned}
$$

- The matrix-vector notation:

$$
\boldsymbol{B} = (b_{ij}), \qquad \boldsymbol{C} = (c_{ij}), \qquad \boldsymbol{x} = (x_i), \qquad \boldsymbol{g} = (g_i), \qquad \boldsymbol{f} = (f_i)
$$

- Representation of the problem in max-plus algebra vector form

$$
\begin{aligned}
\min_{\boldsymbol{x}} \quad & \boldsymbol{x}^{-}\boldsymbol{1}\boldsymbol{1}^{T}\boldsymbol{x}, \\
\text{s.t.} \quad & \boldsymbol{B}\boldsymbol{x} \leq \boldsymbol{x}, \quad \boldsymbol{C}\boldsymbol{x} \leq \boldsymbol{f}, \quad \boldsymbol{g} \leq \boldsymbol{x}
\end{aligned}
$$

Project Scheduling Problem: Algebraic Solution

- Consider the scheduling problem in the max-plus algebra setting

$$\begin{aligned} \min_{\boldsymbol{x}} \quad & \boldsymbol{x}^{-}\boldsymbol{1}\boldsymbol{1}^{T}\boldsymbol{x}, \\ \text{s.t.} \quad & \boldsymbol{B}\boldsymbol{x} \leq \boldsymbol{x}, \quad \boldsymbol{C}\boldsymbol{x} \leq \boldsymbol{f}, \quad \boldsymbol{g} \leq \boldsymbol{x} \end{aligned}$$

- The problem is a special case with $\boldsymbol{A} = \boldsymbol{1}\boldsymbol{1}^{T}$ of the known problem

$$\begin{aligned} \min_{\boldsymbol{x}} \quad & \boldsymbol{x}^{-}\boldsymbol{A}\boldsymbol{x}, \\ \text{s.t.} \quad & \boldsymbol{B}\boldsymbol{x} \leq \boldsymbol{x}, \quad \boldsymbol{C}\boldsymbol{x} \leq \boldsymbol{f}, \quad \boldsymbol{g} \leq \boldsymbol{x} \end{aligned}$$

- We use the solution of the second problem to handle the first one
- We exploit the structure of the objective function in the first problem to simplify conclusive formulas to a large extent

Preliminary Results

- Suppose, given matrices $\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C} \in \mathbb{X}^{n \times n}$ and vectors $\boldsymbol{g}, \boldsymbol{f} \in \mathbb{X}^{n}$, we need to find regular vectors $\boldsymbol{x} \in \mathbb{X}^{n}$ that solve the problem

$$\begin{aligned} \min_{\boldsymbol{x}} \quad & \boldsymbol{x}^{-}\boldsymbol{A}\boldsymbol{x}, \\ \text{s.t.} \quad & \boldsymbol{B}\boldsymbol{x} \leq \boldsymbol{x}, \quad \boldsymbol{C}\boldsymbol{x} \leq \boldsymbol{f}, \quad \boldsymbol{g} \leq \boldsymbol{x} \end{aligned}$$

- To describe a solution of the problem, we introduce the notation

$$\boldsymbol{S}_{k} = \bigoplus_{0 \leq i_1 + \cdots + i_k \leq n-k} \boldsymbol{A}\boldsymbol{B}^{i_1} \cdots \boldsymbol{A}\boldsymbol{B}^{i_k}, \qquad k = 1, \ldots, n;$$

$$\boldsymbol{T}_{k} = \bigoplus_{0 \leq i_0 + i_1 + \cdots + i_k \leq n-k-1} \boldsymbol{B}^{i_0}(\boldsymbol{A}\boldsymbol{B}^{i_1} \cdots \boldsymbol{A}\boldsymbol{B}^{i_k}), \qquad k = 1, \ldots, n-1$$

Problem

$$\min_{\boldsymbol{x}} \ \boldsymbol{x}^{-}\boldsymbol{A}\boldsymbol{x}, \quad \text{s.t.} \ \boldsymbol{B}\boldsymbol{x} \leq \boldsymbol{x}, \ \boldsymbol{C}\boldsymbol{x} \leq \boldsymbol{f}, \ \boldsymbol{g} \leq \boldsymbol{x}$$

Theorem (K. 2014, 2017)

Let $\boldsymbol{A}$ be a matrix with spectral radius $\lambda > \mathbb{0}$.
Let $\boldsymbol{B}$ be a matrix with $\mathrm{Tr}(\boldsymbol{B}) \leq \mathbb{1}$ and $\boldsymbol{C}$ a column-regular matrix.
Let $\boldsymbol{g}$ be a vector and $\boldsymbol{f}$ a regular vector such that $\boldsymbol{f}^{-}\boldsymbol{C}\boldsymbol{B}^{*}\boldsymbol{g} \leq \mathbb{1}$.
Then, the minimum of the objective function in the problem is equal to

$$\theta = \bigoplus_{k=1}^{n} \mathrm{tr}^{1/k}(\boldsymbol{S}_k) \oplus \bigoplus_{k=1}^{n-1} (\boldsymbol{f}^{-}\boldsymbol{C}\boldsymbol{T}_k\boldsymbol{g})^{1/k},$$

and all regular solutions are given in parametric form by

$$\boldsymbol{x} = (\theta^{-1}\boldsymbol{A} \oplus \boldsymbol{B})^{*}\boldsymbol{u}, \qquad \boldsymbol{g} \leq \boldsymbol{u} \leq (\boldsymbol{f}^{-}\boldsymbol{C}(\theta^{-1}\boldsymbol{A} \oplus \boldsymbol{B})^{*})^{-}$$

Solution to Scheduling Problem

Problem

$$\min_{\boldsymbol{x}} \ \boldsymbol{x}^{-}\boldsymbol{1}\boldsymbol{1}^{T}\boldsymbol{x}, \quad \text{s.t.} \ \boldsymbol{B}\boldsymbol{x} \leq \boldsymbol{x}, \ \boldsymbol{C}\boldsymbol{x} \leq \boldsymbol{f}, \ \boldsymbol{g} \leq \boldsymbol{x}$$

Lemma

Let $\boldsymbol{B}$ be a matrix with $\mathrm{Tr}(\boldsymbol{B}) \leq \mathbb{1}$ and $\boldsymbol{C}$ a column-regular matrix. Let $\boldsymbol{g}$ be a vector and $\boldsymbol{f}$ a regular vector such that $\boldsymbol{f}^{-}\boldsymbol{C}\boldsymbol{B}^{*}\boldsymbol{g} \leq \mathbb{1}$. Then, the minimum of the objective function in the problem is equal to

$$\theta = \|\boldsymbol{B}^{*}\| \oplus \bigoplus_{0 \leq i+j \leq n-2} \|\boldsymbol{f}^{-}\boldsymbol{C}\boldsymbol{B}^{i}\|\|\boldsymbol{B}^{j}\boldsymbol{g}\|,$$

and all regular solutions are given in parametric form by

$$\boldsymbol{x} = \boldsymbol{G}\boldsymbol{u}, \qquad \boldsymbol{g} \leq \boldsymbol{u} \leq (\boldsymbol{f}^{-}\boldsymbol{C}\boldsymbol{G})^{-}, \qquad \boldsymbol{G} = \boldsymbol{B}^{*} \oplus \bigoplus_{0 \leq i+j \leq n-2} \theta^{-1}\boldsymbol{B}^{i}\boldsymbol{1}\boldsymbol{1}^{T}\boldsymbol{B}^{j}$$

Sketch of Proof: Substitution

- To prove the Lemma, we need to substitute $\boldsymbol{A} = \boldsymbol{1}\boldsymbol{1}^T$ in the solution provided by Theorem and then simplify the result
- By Theorem, the minimum of the objective function is equal to

$$\theta = \bigoplus_{k=1}^{n} \mathrm{tr}^{1/k}(\boldsymbol{S}_k) \oplus \bigoplus_{k=1}^{n-1} (\boldsymbol{f}^{-}\boldsymbol{C}\boldsymbol{T}_k\boldsymbol{g})^{1/k},$$

where the sums $\boldsymbol{S}_k$ and $\boldsymbol{T}_k$ after substitution $\boldsymbol{A} = \boldsymbol{1}\boldsymbol{1}^T$ become

$$\boldsymbol{S}_k = \bigoplus_{0 \leq i_1 + \cdots + i_k \leq n-k} \boldsymbol{1}\boldsymbol{1}^T\boldsymbol{B}^{i_1} \cdots \boldsymbol{1}\boldsymbol{1}^T\boldsymbol{B}^{i_k}, \qquad k = 1, \dots, n;$$

$$\boldsymbol{T}_k = \bigoplus_{0 \leq i_0 + i_1 + \cdots + i_k \leq n-k-1} \boldsymbol{B}^{i_0}(\boldsymbol{1}\boldsymbol{1}^T\boldsymbol{B}^{i_1} \cdots \boldsymbol{1}\boldsymbol{1}^T\boldsymbol{B}^{i_k}), \qquad k = 1, \dots, n-1$$

- All regular solutions are given by the conditions

$$\boldsymbol{x} = (\theta^{-1}\boldsymbol{1}\boldsymbol{1}^T \oplus \boldsymbol{B})^{*}\boldsymbol{u}, \qquad \boldsymbol{g} \leq \boldsymbol{u} \leq (\boldsymbol{f}^{-}\boldsymbol{C}(\theta^{-1}\boldsymbol{1}\boldsymbol{1}^T \oplus \boldsymbol{B})^{*})^{-}$$

Sketch of Proof: Evaluation of Minimum

- The equality $\mathbf{1}^T\boldsymbol{B}\mathbf{1} = \|\boldsymbol{B}\|$ and the cyclic property of trace yield

$$\bigoplus_{k=1}^{n} \operatorname{tr}^{1/k}(\boldsymbol{S}_k) = \bigoplus_{k=1}^{n} \bigoplus_{0\leq i_1+\cdots+i_k\leq n-k} (\|\boldsymbol{B}^{i_1}\|\cdots\|\boldsymbol{B}^{i_k}\|)^{1/k}$$

- The inequality between geometric and arithmetic means leads to

$$(\|\boldsymbol{B}^{i_1}\|\cdots\|\boldsymbol{B}^{i_k}\|)^{1/k} \leq \|\boldsymbol{B}^{i_1}\|\oplus\cdots\oplus\|\boldsymbol{B}^{i_k}\| \leq \bigoplus_{i=0}^{n-1}\|\boldsymbol{B}^{i}\|$$

- Application of the last inequality results in the double inequality

$$\bigoplus_{i=0}^{n-1}\|\boldsymbol{B}^{i}\| = \operatorname{tr}\boldsymbol{S}_1 \leq \bigoplus_{k=1}^{n} \operatorname{tr}^{1/k}(\boldsymbol{S}_k) \leq \bigoplus_{i=0}^{n-1}\|\boldsymbol{B}^{i}\| = \operatorname{tr}\boldsymbol{S}_1$$

- Since $\operatorname{tr}\boldsymbol{S}_1$ dominates the other terms, the first sum reduces to

$$\bigoplus_{k=1}^{n} \operatorname{tr}^{1/k}(\boldsymbol{S}_k) = \operatorname{tr}\boldsymbol{S}_1 = \bigoplus_{i=0}^{n-1}\|\boldsymbol{B}^{i}\| = \|\boldsymbol{B}^{*}\|$$

Sketch of Proof: Evaluation of Minimum

- With the matrix/vector norm notation, the second sum becomes

$$\bigoplus_{k=1}^{n-1}(\boldsymbol{f}^{-}\boldsymbol{C}\boldsymbol{T}_k\boldsymbol{g})^{1/k} = \bigoplus_{k=1}^{n-1} \bigoplus_{0\leq i_0+i_1+\cdots+i_k\leq n-k-1} (\|\boldsymbol{B}^{i_1}\|\cdots\|\boldsymbol{B}^{i_{k-1}}\|)^{1/k} \otimes \|\boldsymbol{f}^{-}\boldsymbol{C}\boldsymbol{B}^{i_0}\|^{1/k}\|\boldsymbol{B}^{i_k}\boldsymbol{g}\|^{1/k}$$

- Using similar arguments as before leads to the double inequality

$$\boldsymbol{f}^{-}\boldsymbol{C}\boldsymbol{T}_1\boldsymbol{g} \leq \bigoplus_{k=1}^{n-1}(\boldsymbol{f}^{-}\boldsymbol{C}\boldsymbol{T}_k\boldsymbol{g})^{1/k} \leq \operatorname{tr}(\boldsymbol{S}_1) \oplus \boldsymbol{f}^{-}\boldsymbol{C}\boldsymbol{T}_1\boldsymbol{g}$$

- Combining the results yields the minimum in the problem, given by

$$\theta = \bigoplus_{k=1}^{n}\operatorname{tr}^{1/k}(\boldsymbol{S}_k)\oplus\bigoplus_{k=1}^{n-1}(\boldsymbol{f}^{-}\boldsymbol{C}\boldsymbol{T}_k\boldsymbol{g})^{1/k} = \|\boldsymbol{B}^{*}\|\oplus\bigoplus_{0\leq i+j\leq n-2}\|\boldsymbol{f}^{-}\boldsymbol{C}\boldsymbol{B}^{i}\|\|\boldsymbol{B}^{j}\boldsymbol{g}\|$$

Sketch of Proof: Representation of Solutions

- It follows from Theorem that all solution vectors are given by

$$\boldsymbol{x} = (\theta^{-1}\mathbf{1}\mathbf{1}^T \oplus \boldsymbol{B})^*\boldsymbol{u}, \qquad \boldsymbol{g} \leq \boldsymbol{u} \leq (\boldsymbol{f}^-\boldsymbol{C}(\theta^{-1}\mathbf{1}\mathbf{1}^T \oplus \boldsymbol{B})^*)^-$$

- Specifically, the Kleene star matrix takes the form

$$(\theta^{-1}\mathbf{1}\mathbf{1}^T \oplus \boldsymbol{B})^* = \boldsymbol{B}^* \oplus \bigoplus_{k=1}^{n-1} \bigoplus_{m=0}^{n-k-1} \bigoplus_{i_0+\dots+i_k=m} \theta^{-m}\boldsymbol{B}^{i_0}\mathbf{1}\mathbf{1}^T\boldsymbol{B}^{i_k}(\|\boldsymbol{B}^{i_1}\|\cdots\|\boldsymbol{B}^{i_{k-1}}\|)$$

- Similarly as before, the second sum is reduced, which results in

$$(\theta^{-1}\mathbf{1}\mathbf{1}^T \oplus \boldsymbol{B})^* = \boldsymbol{B}^* \oplus \bigoplus_{0\leq i+j\leq n-2} \theta^{-1}\boldsymbol{B}^i\mathbf{1}\mathbf{1}^T\boldsymbol{B}^j = \boldsymbol{G}$$

Summary of Obtained Solution

Problem

$$\min_{\boldsymbol{x}} \ \boldsymbol{x}^{-}\mathbf{1}\mathbf{1}^{T}\boldsymbol{x}, \quad \text{s.t.} \ \boldsymbol{B}\boldsymbol{x} \leq \boldsymbol{x}, \ \boldsymbol{C}\boldsymbol{x} \leq \boldsymbol{f}, \ \boldsymbol{g} \leq \boldsymbol{x}$$

Lemma

Let $\boldsymbol{B}$ be a matrix with $\mathrm{Tr}(\boldsymbol{B}) \leq \mathbb{1}$ and $\boldsymbol{C}$ a column-regular matrix. Let $\boldsymbol{g}$ be a vector and $\boldsymbol{f}$ a regular vector such that $\boldsymbol{f}^{-}\boldsymbol{C}\boldsymbol{B}^{*}\boldsymbol{g} \leq \mathbb{1}$. Then, the minimum of the objective function in the problem is equal to

$$\theta = \|\boldsymbol{B}^{*}\| \oplus \bigoplus_{0 \leq i+j \leq n-2} \|\boldsymbol{f}^{-}\boldsymbol{C}\boldsymbol{B}^{i}\|\|\boldsymbol{B}^{j}\boldsymbol{g}\|,$$

and all regular solutions are given in parametric form by

$$\boldsymbol{x} = \boldsymbol{G}\boldsymbol{u}, \qquad \boldsymbol{g} \leq \boldsymbol{u} \leq (\boldsymbol{f}^{-}\boldsymbol{C}\boldsymbol{G})^{-}, \qquad \boldsymbol{G} = \boldsymbol{B}^{*} \oplus \bigoplus_{0 \leq i+j \leq n-2} \theta^{-1}\boldsymbol{B}^{i}\mathbf{1}\mathbf{1}^{T}\boldsymbol{B}^{j}$$

Numerical Example

- Consider a project with $n = 4$ activities, where the start-start and start-finish constraints are given by the matrices

$$
\boldsymbol{B} = \begin{pmatrix} 0 & -1 & \mathbb{0} & \mathbb{0} \\ 1 & 0 & 0 & \mathbb{0} \\ -2 & -1 & 0 & \mathbb{0} \\ 0 & -3 & 2 & 0 \end{pmatrix}, \qquad \boldsymbol{C} = \begin{pmatrix} 4 & 0 & \mathbb{0} & \mathbb{0} \\ 1 & 3 & \mathbb{0} & 2 \\ 4 & \mathbb{0} & 5 & 0 \\ 5 & 1 & 4 & 3 \end{pmatrix},
$$

and the release time and deadline constraints by the vectors

$$
\boldsymbol{g} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \qquad \boldsymbol{f} = \begin{pmatrix} 6 \\ 6 \\ 6 \\ 6 \end{pmatrix}
$$

Numerical Example (cont.)

- The evaluation of the minimum deviation of start times gives

$$\theta = \|\boldsymbol{B}^{*}\| \oplus \|\boldsymbol{f}^{-}\boldsymbol{C}\|\|\boldsymbol{g}\| \oplus \|\boldsymbol{f}^{-}\boldsymbol{C}\boldsymbol{B}\|\|\boldsymbol{g}\| \oplus \|\boldsymbol{f}^{-}\boldsymbol{C}\|\|\boldsymbol{B}\boldsymbol{g}\| \oplus \|\boldsymbol{f}^{-}\boldsymbol{C}\boldsymbol{B}^{2}\|\|\boldsymbol{g}\| \oplus \|\boldsymbol{f}^{-}\boldsymbol{C}\boldsymbol{B}\|\|\boldsymbol{B}\boldsymbol{g}\| \oplus \|\boldsymbol{f}^{-}\boldsymbol{C}\|\|\boldsymbol{B}^{2}\boldsymbol{g}\| = 2$$

- Next, we calculate the Kleene matrix and the boundary vector

$$\boldsymbol{G} = \boldsymbol{B}^{*} \oplus (-2)(\boldsymbol{1}\boldsymbol{1}^{T} \oplus \boldsymbol{B}\boldsymbol{1}\boldsymbol{1}^{T} \oplus \boldsymbol{1}\boldsymbol{1}^{T}\boldsymbol{B} \oplus \boldsymbol{B}^{2}\boldsymbol{1}\boldsymbol{1}^{T} \oplus \boldsymbol{B}\boldsymbol{1}\boldsymbol{1}^{T}\boldsymbol{B} \oplus \boldsymbol{1}\boldsymbol{1}^{T}\boldsymbol{B}^{2})$$

$$= \begin{pmatrix} 0 & -1 & 0 & -2 \\ 1 & 0 & 1 & -1 \\ 0 & -1 & 0 & -2 \\ 2 & 1 & 2 & 0 \end{pmatrix}, \qquad (\boldsymbol{f}^{-}\boldsymbol{C}\boldsymbol{G})^{-} = \begin{pmatrix} 1 \\ 2 \\ 1 \\ 3 \end{pmatrix}$$

Numerical Example (cont.)

- The optimal start times of activities are given in parametric form by

$$\boldsymbol{x} = \begin{pmatrix} 0 & -1 & 0 & -2 \\ 1 & 0 & 1 & -1 \\ 0 & -1 & 0 & -2 \\ 2 & 1 & 2 & 0 \end{pmatrix} \boldsymbol{u}, \qquad \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \leq \boldsymbol{u} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix} \leq \begin{pmatrix} 1 \\ 2 \\ 1 \\ 3 \end{pmatrix}$$

- All columns in the matrix $\boldsymbol{G}$ are collinear (in the max-plus algebra sense) and hence the matrix can be factorized as follows:

$$\boldsymbol{G} = \begin{pmatrix} 0 & -1 & 0 & -2 \\ 1 & 0 & 1 & -1 \\ 0 & -1 & 0 & -2 \\ 2 & 1 & 2 & 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 2 \end{pmatrix} \begin{pmatrix} 0 & -1 & 0 & -2 \end{pmatrix}$$

Numerical Example (cont.)

- The vector of parameters $\boldsymbol{u}$ can now be replaced by the scalar

$$u = u_1 \oplus (-1)u_2 \oplus u_3 \oplus (-2)u_4$$

- With the new scalar parameter, the solution takes the form

$$\boldsymbol{x} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 2 \end{pmatrix} u, \qquad 0 \leq u \leq 1$$

- In terms of conventional algebra, the solution is represented as

$$x_1 = u, \quad x_2 = u + 1, \quad x_3 = u, \quad x_4 = u + 2, \qquad 0 \leq u \leq 1$$