On Morse index recovering

Daniil Mamaev and Gaiane Panina

Imagine there is a smooth real-valued function f defined in a ball B^d with a unique critical point at the center of the ball O. Assume that the critical point O is a Morse one, and hence possesses the Morse index. We shall discuss the following problem (and arrive at a somewhat counterintuitive solution):
"Up to what extent can one recover the Morse index M by the behaviour of the function in the ball with a deleted neighborhood of the center $B^d \setminus \epsilon B^d$?"

Some simple observations are: (1) In certain cases the Morse index is retrieved uniquely. For instance, if the gradient of f points outwards at each point at the boundary sphere ∂B^d, then clearly O is a minimum point, that is, $M = 0$. (2) The direction of the gradient field on the sphere ∂B^d defines a map $\partial B^d \longrightarrow \partial B^d$ whose index is ± 1 depending on the parity of M. Therefore, the parity of M is retrievable.

Our main conjecture that in the general case, nothing but the parity of M can be retrieved. We shall prove the following

Theorem.
For any d and M such that $0 \leq M, M+2 \leq d$ there exist two smooth functions $f,\ g : B^d \longrightarrow \mathbb{R}$ such that

1. $f \equiv g$ *on* $B^d \setminus \epsilon B^d$.
2. *Both f and g have a unique critical point at O, and the Morse indices are M and $M+2$ respectively.*

One of the key ideas was to use the Morse-Barannikov framed complex which provided the combinatorial counterpart of the construction.

Daniil Mamaev
Chebyshev Laboratory at St. Petersburg University
e-mail: dan.mamaev@gmail.com

Gaiane Panina
St. Petersburg Department of Steklov Institute of Mathematics
e-mail: gaiane-panina@rambler.ru