Theory of Proofs, Categorical Semantics and Non-Free Categories.

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Plan

- Introduction.
- Sequential Calculus L(A) for Intuitionistic Multiplicative Linear Logic.
- Categorical Models and Interpretations
- Axiomatizations (by critical pairs) and Varieties
- Models (more or less exotic)
- Triple-Dual Conjecture
- Intermediate [Categorical] Equivalences on L(A)
- Arbitrary Natural Transformations

Introduction

Usually when categorical semantics of proofs is considered it is based on the structure of the free category of a certain type defined on a certain logical system:

- free CCC: $(\wedge, \rightarrow)$-fragment of the IPC;
- free SMCC: $(\otimes, \multimap)$-fragment of the IMLL.

In fact the list of logical systems and associated categorical structures is quite long (cf. “Cut elimination in categories” by K. Dosen [Dos])

- **Why non-free categories are usually not considered?**
- Erroneous intuition: syntactic structure is too poor.

Introduction

Two-directional relation:

- Categorical structure on a proof-system permits to use proof theory (theory of proofs) to study other (non-free) categories.
- New structures on proof-systems (e.g., new equivalence relations on derivations) may be "induced" by some features of non-free categories .

Introduction

In this talk I will consider the following problems:

- Equivalences on derivations generated by interpretations in non-free categories. Different types of equivalences, critical pairs and canonical axiomatization.
- Triple-dual conjecture and the problem of full coherence in closed categories.
- Varieties of categories with structure.
- Commutativity and dependency of diagrams in non-free categories. Proof-theoretic methods of verification of commutativity.
- Arbitrary natural transformations and proof-theoretic methods in their study.

Introduction

- As a main source of examples:
- **Mostly will be considered Symmetric Monoidal Closed Categories (SMCC)**
- **and the related logical system - Intuionistic Multiplicative Linear Logic (IMLL)**
- Represented by the sequent calculus **L(A).**
- Many algebraic examples.

L(A) and interpretations

The calculus **L**(**A**):

Axioms

$A \to A \quad (1_A) \qquad \to I \quad (\text{unit})$

Logical rules

$$\frac{\Gamma \to A \quad \Delta \to B}{\Gamma, \Delta \to A \otimes B} \ (\to\otimes) \qquad \frac{A, B, \Gamma \to C}{A \otimes B, \Gamma \to C} (\otimes\to)$$

$$\frac{A, \Gamma \to B}{\Gamma \to A \multimap B} (\to\multimap) \qquad \frac{\Gamma \to A \quad B, \Delta \to C}{\Gamma, A \multimap B, \Delta \to C} (\multimap\to)$$

L(A) and interpretations

Structural Rules

$$\frac{\Gamma \to A \quad A, \Delta \to B}{\Gamma, \Delta \to B}\text{(cut)} \qquad \frac{\Delta \to I \quad \Sigma \to A}{\Delta, \Sigma \to A}\text{(wkn)}$$

$$\frac{\Gamma \to A}{\Gamma' \to A}\text{(perm)}$$

L(A) and interpretations

- On $\mathbf{L}(\mathbf{A})$ a structure of a free SMCC over the set of atoms **A** can be defined.
- In this structure the equivalence classes of derivations of the sequent $A \rightarrow B$ play the role of morphisms from A to B.
- The equivalence of free SMCC on derivations will be denoted by $\equiv$. (It is defined as the smallest equivalence relation satisfying certain axioms.)
- **Remark.** This schema works for many of the calculi defined in a similar way.

L(A) and interpretations

- As for every free category, any valuation $v : \mathbf{A} \to Ob(K)$ where K is some SMCC defines a unique structure-preserving functor $| - |_v : \mathbf{L}(\mathbf{A}) \to K$.
- Via this functor a new equivalence relation $\sim_v$ on derivations can be defined: $d \sim_v d' \iff |d|_v = |d'|_v$. The relation $\sim_v$ also defines a certain structure of SMCC on $\mathbf{L}(\mathbf{A})$; $\equiv \subseteq \sim_v$.
- **Remark.**The relation $\sim_v$ is a congruence with respect to the rules of $\mathbf{L}(\mathbf{A})$, but in general it is not closed with respect to substitution of formulas for variables.

L(A) and interpretations

- Relations $\sim$ that define the structure of SMCC on $\mathbf{L}(\mathbf{A})$ have many interesting properties but we shall mostly consider two types of relations. Let us fix some SMCC K:
- (a) $d \sim_\forall d' \iff \forall v : \mathbf{A} \to Ob(K).(d \sim_v d')$.
- (b) Let $NAT(K)$ be the SMCC category of functors and natural transformations over K. Let v_1 denote the interpretation of $\mathbf{L}(\mathbf{A})$ in $NAT(K)$ defined by $a \mapsto 1 : K \to K$ (every $a \in \mathbf{A}$ has the identity functor $1 : K \to K$ as its value). Now $d \sim_{nat} d' \iff .d \sim_{v_1} d'$.
- The relations $\sim_{nat}$ and $\sim_\forall$ are both congruences with respect to the rules of $\mathbf{L}(\mathbf{A})$. Both are closed with respect to substitution of formulas for atoms.

L(A) and interpretations

Let us mention a general result (see [MS]) concerning these categorical relations on derivations of $\mathbf{L}(\mathbf{A})$.

Theorem

Let K be an SMCC category with biproduct. Then the relations $\sim_{nat}$ and $\sim_{\forall}$ (defined with respect to K) coincide.

Why? It is possible to "diversify" variables: substitute $x_{i1} \oplus ... \oplus x_{in}$ for each variable that has n covariant and n contravariant occurrences and use naturality conditions. ("Balancedness" may be shown.)

$$x \otimes x \to x \otimes x \mapsto (x_1 \oplus x_2) \otimes (x_1 \oplus x_2) \to (x_1 \oplus x_2) \otimes (x_1 \oplus x_2)$$

Depending on naturality conditions either $x_1 \otimes x_2 \to x_1 \otimes x_2$ or $x_1 \otimes x_2 \to x_2 \otimes x_1$ is obtained.

L(A) and interpretations

The relation $\equiv$ is defined on the derivations of $\mathbf{L}(\mathbf{A})$ by the axioms of SMCC. Clearly, all the relations $\sim$ that define some SMCC structure on $\mathbf{L}(\mathbf{A})$ are defined by adding some new axioms (equivalences between derivations) to the axioms of SMCC. The same relations can be defined via $|-|_v$ for some K and v (if necessary, one may take the factor category of $\mathbf{L}(\mathbf{A})$ by $\sim$ as K).

A non-trivial problem here is to find some canonical axiomatization of these relations.

Axiomatizations

Easy results:

Definition

A sequent S is called balanced if every atom has exactly two occurrences with opposite signs in S. It is pure if there are no subformulas of the form $I \otimes A$, $A \otimes I$, $I \multimap A$.

Theorem

A relation of type $\sim_{nat}$ can always be axiomatized using the axioms of the form $d \sim d' : \Gamma \to A$ where $\Gamma \to A$ is balanced and pure.

Axiomatizations

A useful equivalence-preserving transformation (cf. [Sol87, Sol90, Sol97, MS]):

Definition

The sequent $\Gamma \to A$ is called a 2-sequent if A contains no more than one connective and each member of Γ no more than two connectives.

Some formulas may be replaced by isomorphic ones (reducing further the number of possibilities).

Definition

$\Gamma \to A$ is called a pure 2-sequent if A has one of the forms $x, a \otimes b, a \multimap x$ and each member of Γ has one of the forms $x, a \multimap x, a \multimap (b \otimes c), (a \otimes b) \multimap x, (a \multimap x) \multimap y$. Here x, y stand for I or atoms, a, b are atoms.

Axiomatizations

Remark. Reduction is based on two operations (followed, if needed, by isomorphisms to obtain a *pure* 2-sequent):

- $$\Gamma \xrightarrow{d} B \mapsto \frac{\Gamma \xrightarrow{d} B \quad p \xrightarrow{id} p}{\Gamma, B \multimap p \to p} (p\ fresh)$$
- *cut* with $p \multimap C, A[p] \to A[C]$
  or with $C \multimap p, A[p] \to A[C]$
  where variable p is fresh.
- For example

$$\frac{A \multimap p, p \multimap B \to A \multimap B \quad \Gamma, A \multimap B \xrightarrow{d} D}{\Gamma, A \multimap p, p \multimap B \to D}$$

Axiomatizations

Theorem

(Reduction to 2-sequents.) Let an equivalence relation $\sim$ *on derivations contain* $\equiv$ *and be a substitutive congruence. Let* d_1, d_2 *be two derivations of the same sequent* S*. Then there exist two derivations* d'_1, d'_2 *of the same pure 2-sequent* S' *(balanced if* S *was balanced) such that* d_1, d_2 *are* $\sim$*-equivalent iff* d'_1, d'_2 *are* $\sim$*-equivalent.*

Remark. Generalizations are possible!
Remark. In SMCC with additional (stronger) properties, stronger reductions may be possible. For exemple in Compact Closed (where $(A \multimap I) \multimap I$ is naturally isomorphic to A and $A \multimap B$ to $(A \multimap I) \otimes B$) "flattening" is possible (reduction to the sequents containing only $\otimes$).

Axiomatizations

Definition

A pair of derivations of a balanced pure 2-sequent S is critical if

(1) $d_1 \equiv \dfrac{\Gamma, A' \multimap I \overset{d_1'}{\to} A \quad I \to I}{\Gamma, A' \multimap I, A \multimap I \to I} \multimap \to,$

$d_2 \equiv \dfrac{\Gamma, A \multimap I \overset{d_2'}{\to} A' \quad I \to I}{\Gamma, A' \multimap I, A \multimap I \to I} \multimap \to, perm;$

(2) a *cut*-free derivation of S can end only by some application of $\multimap \to$;

(3) d_1', d_2' are not $\equiv$-equivalent to derivations ending by $\multimap \to$.

The pair is minimal if Γ does not contain members being single atoms.

Axiomatizations

- Let α be some substitution of I for variables. In [Sol97] the "substitutions with purification" were defined: $\alpha * d$ is the derivation obtained from d by α and *cuts* with isomorphisms that will make its final sequent pure. The derivation $\alpha * d$ is defined up to $\equiv$, but its final sequent is defined without ambiguity.

Theorem

(Cf.[Sol97].) Let d_1, d_2 be derivations of a balanced sequent $\Gamma \to A$ and d_1', d_2' the corresponding derivations of a balanced pure 2-sequent. Then $d_1 \equiv d_2$ iff there exists a substitution α of I for variables such that $\alpha * d_1', \alpha * d_2'$ is a minimal critical pair.

Axiomatizations

The following theorem shows that every relation of the type $\sim_{nat}$ is generated (i.e., **may be axiomatized**) by minimal critical pairs.

Theorem (MS)

For every relation $\sim_{nat}$ there exists some set M of minimal critical pairs such that $\sim_{nat}$ is the smallest equivalence relation that is a congruence with respect to the rules of $\mathbf{L}(\mathbf{A})$, closed with respect to substitution, contains $\equiv$ and all the pairs $(d, d') \in M$.

Models

- Main theorem of [Sol97] established that any category of vector spaces over a field is a complete model of the theory of SMCC.
- In fact - also any category of modules over a commutative ring,
- and even for the category of finite pointed sets.
- (There is a list of sufficient conditions.)
- The theorem about critical pairs is used to show that in appropriate models one may find a valuation that non-commutative diagram remains non-commutative.

Models

- **The diversity of models may be very important for the proof of the triple-dual conjecture that will be discussed below.**
- One class of models I would like to mention is that of game-theoretic models.
- A *-autonomous category is an SMCC with a dualizing object $\bot$, i.e., an object such that for every A, A and $(A \multimap \bot) \multimap \bot$ are naturally isomorphic.
- "SMCC of sequential games are *nearly*-*-autonomous. There exists... a tentative *dualizing* object in these categories: the sequential game $\bot$ with a unique move played by Opponent... There exists a retraction $(A \multimap \bot) \multimap \bot \to A$..." [HAL]

Triple-dual conjecture

The problem of full coherence in closed categories and the triple-dual conjecture.
Below we shall call diagrams any pairs of **L(A)**-derivations of arbitrary sequent $\Gamma \to A$.
A sequent S is called proper if it does not contain occurrences of subformulas of the form $A \multimap B$ where B is constant (contains only I) and A is not constant.

Theorem

(The Kelly-Mac Lane coherence theorem reformulated for **L(A)***, cf.[KM].) Let $f, g : \Gamma \to A$ and assume that the sequent $\Gamma \to A$ is proper. If f and g have the same graph*[a] *then $f \equiv g$.*

[a] In particular, if the sequent is balanced.

Triple-dual conjecture

Example

The following diagram (called “triple-dual” diagram) is non-commutative

$$
(1) \qquad
\begin{array}{ccc}
((a \multimap I) \multimap I) \multimap I & \xrightarrow{\quad 1 \quad} & ((a \multimap I) \multimap I) \multimap I \\
 & {\scriptstyle k_{a \multimap 1}} \searrow \qquad \nearrow {\scriptstyle k_{a \multimap I}} & \\
 & a \multimap I &
\end{array}
$$

where a is a variable, $k_a = (1 \multimap e_{aI}) \circ d_{a(a \multimap I)} : a \to (a \multimap I) \multimap I$ is the standard “embedding of a into its second dual”.

Triple-dual conjecture

- Non-commutativity of this diagram may be checked formally (the equivalence relation $\equiv$ is decidable).
- It is non-commutative also in certain common models such as the SMCC of vector spaces or the SMCC of modules over a commutative ring with unit.
- One may note that it is always commutative in the full subcategory of vector spaces of finite dimension.
- To the contrary, it is not always commutative for finitely generated modules - we consider an example below.

Triple-dual conjecture

CONJECTURE.

- Commutativity of the triple-dual diagram implies commutativity of all diagrams of canonical maps $f, g : A \to B$ with balanced $A \to B$.
- More precisely: let $\sim$ be the smallest equivalence relation that satisfies all axioms of SMCC, is substitutive and the triple-dual diagram is commutative w.r.t. $\sim$. Then for all $f, g : A \to B$ with balanced $A \to B$ $f \sim g$.

Triple-dual conjecture

An argument in favor of this conjecture is that with just one additional condition

Condition. If f is NOT $\sim$-equivalent to g then $[a \multimap I/a]f$ is NOT $\sim$-equivalent to $[a \multimap I/a]g$

one has the following theorem:

Theorem

(Soloviev, 1990 [Sol90].) If $\sim$ is the smallest equivalence relation that satisfies all axioms of SMCC, is substitutive, the triple-dual diagram is commutative w.r.t. $\sim$, and $\sim$ satisfies the condition above then for all $f, g : A \to B$ with balanced $A \to B$ $f \sim g$.

This conjecture is still open but there ARE other equivalence relations different from $\equiv$. (All known equivalence relations are between $\equiv$ and $\sim$.)

Triple-dual conjecture

- It will be useful to consider another diagram which is commutative iff the triple-dual diagram is commutative:

$$(2)\ \ f, g:\ (a \otimes b \multimap I), ((b \multimap I) \multimap I), ((a \multimap I) \multimap I) \to I.$$

- In his thesis Antoine El Khoury (ph.d. student of S. Soloviev and M. Spivakovsky) checked that the commutativity of the triple-dual diagram implies the commutativity of all diagrams $f, g : A \to B$ with balanced $A \to B$ containing no more than 3 variables.

Intermediate equivalences

- The diagram (2) may be used also to obtain intermediate equivalence relations mentioned above.
- **We build some models where (2) (or some “derived” diagram) is non-commutative but some other is commutative.**
- I had an opportunity to speak about this result at one of previous PCA.

Intermediate equivalences

(a) Model suggested by M. Spivakovsky and developped by L. Méhats.

The diagram may be obtained from

$$(2)\ \ f, g :\ (a \otimes b \multimap I), ((b \multimap I) \multimap I), ((a \multimap I) \multimap I) \to I$$

by cut with

$$\varphi : ((a \multimap I) \otimes (b \multimap I) \multimap I) \multimap I \to (a \otimes b \multimap I)$$

The result is

(3) $f_1, g_1\ :\ , ((a \multimap I) \otimes (b \multimap I) \multimap I) \multimap I,$
$((b \multimap I) \multimap I), ((a \multimap I) \multimap I) \to I$

Intermediate equivalences

- The model is a subcategory of the category of modules over a ring where (3) is commutative and (2) is not.
- **For example, SMCC generated by** $I = Z_2[x, y]/(x^2, xy, y^2), M = Z_2$
- So, the commutativity of (3) does not imply the commutativity of (2) and (1), and it gives a $\sim_n$ *at* between $\equiv$ and $\sim$.
- In fact, in this category φ is 0 when a, b are not isomorphic to I, and if a or b is isomorphic to I, then (2) is commutative.

Intermediate equivalences

- In our joint work we have shown that there exists an infinite series of the relations of type $\sim_\forall$.
- It was obtained an infinite series of diagrams $\{D_n\}$ such that there are models where $D_1, ..., D_n$ are non-commutative but $D_{n+1}, ...$ are commutative.
- **We used as models the SMCCs of (certain special) commutative semi-modules over commutative semi-rings**
- **Remark.** To be explored: more "exotic" models, as game-theoretic (cf. [HAL]). Unfortunately the models mentioned above were not enough to **prove** (or **disprove**) the **triple-dual conjecture.**

Intermediate equivalences

- **A few words how all this is related to varieties of SMCCs and dependency of diagrams...**
- Varieties: axiomatization by identities, i.e., commutative diagrams.
- Dependency: (D2) depends on (D1) if (D2) commutes in all models where (D1) commutes.
- **Now, as promised, something will be said about arbitrary natural transformations.**

Arbitrary NT

There is the following key lemma:(Cf.[Sol87].)

Lemma

Let K be an SMCC, and I be a generator in K. Let $\phi : |(A \multimap p) \otimes (p \multimap B)| \to |C|$ be a natural transformation in K (the variable p has only two occurrences shown explicitely). Then ϕ is equal to the following composition:

$$|(A \multimap p) \otimes (p \multimap B)| \stackrel{|Comp|}{\to} |A \multimap B| \stackrel{\phi_0}{\to} |C|$$

where $Comp : (A \multimap p) \otimes (p \multimap B) \to A \multimap B$ is a standard derivation representing composition, and ϕ_0 is some natural transformation in K that has one argument less than ϕ.

Arbitrary NT

What are the consequences?

- **(a) Full description in case of Compact Closed Categories where I is a generator.**
- **(b) Full description in case of $\otimes$ and BIPRODUCT. (R. Cockett, S. Soloviev, M. Hyland, 2001 [CHS].**
- Cf. also [Sol20], [Zekic]. (Compact Closed Categories with Biproduct.)

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