On unique tensor decompositions

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Abstract. Kruskal's theorem states that a sum of product tensors constitutes a unique tensor rank decomposition if the so-called k-ranks of the product tensors are large. We prove a more general result in which the k-rank condition of Kruskal's theorem is weakened to the standard notion of rank, and the conclusion is relaxed to a statement on the linear dependence of the product tensors. As a corollary, we prove that if n product tensors form a circuit, then they have rank greater than one in at most $n-2$ subsystems. This generalizes several recent results in this direction, and is sharp. The proof of the main result is based on the matroid ear decomposition technique.

For a non-negative integer n we write $[n] = \{1, 2, \dots, n\}$.

Let $\mathbb{F}$ be a field.

Let $\mathcal{U}$ be a vector space over $\mathbb{F}$ and let $E = \{e_1, \dots, e_n\} \subset \mathcal{U}$ be a finite multiset.

We say that E is *a circuit*, if all n elements of E are linearly dependent, but any $n-1$ of them are linearly independent (this is a matroid theory concept).

We say that E *splits*, if there exists a partition $[n] = J_1 \sqcup J_2$ such that J_1, J_2 are non-empty and

$$\operatorname{span}\{e_i : i \in J_1\} \cap \operatorname{span}\{e_i : i \in J_2\} = \{0\}.$$

In other words, E splits if it is disconnected as a matroid.

Further, let $m > 1$ an integer, let $\mathcal{V}_1, \dots, \mathcal{V}_m$ be vector spaces over $\mathbb{F}$. Further we refer to their tensor product $\mathcal{V} = \mathcal{V}_1 \otimes \cdots \otimes \mathcal{V}_m$ as a *multipartite vector space*. A *product tensor* in $\mathcal{V}$ is a non-zero tensor $z \in \mathcal{V}$ of the form $z = z_1 \otimes \cdots \otimes z_m$, with $z_j \in \mathcal{V}_j$ for all $j \in [m]$. We refer to the spaces $\mathcal{V}_j$ that make up the space $\mathcal{V}$ as *subsystems*. The *tensor rank* (or *rank*) of a tensor $v \in \mathcal{V}$, denoted by $\operatorname{rank}(v)$, is the minimum number r for which v is the sum of r product tensors. A decomposition of v into the sum of r product tensors is called a *tensor rank decomposition* of v.

A *uniqueness* of a decomposition of a tensor x as a sum of n product tensors is understood naturally (up to permuting the summands). If a decomposition of the tensor x as a sum of $n = \operatorname{rank}(x)$ product tensors is unique, it is called the *unique tensor rank decomposition*.

Recall the classical sufficient condition of the uniqueness:

Theorem 1. *Let $n \geqslant 2$ and $m \geqslant 3$ be integers, $\mathcal{V} = \mathcal{V}_1 \otimes \cdots \otimes \mathcal{V}_m$ be a multipartite vector space. Let*

$$E = \{x_{a,1} \otimes \cdots \otimes x_{a,m} : a \in [n]\} \subseteq \mathcal{V} \tag{1}$$

be a multiset of product tensors. Assume that positive integers k_j, $j \in [m]$, be such that any k_j vectors in the set $\{x_{1,j}, \dots, x_{n,j}\}$ are linearly independent. If $2n \leqslant 1 + \sum_{j=1}^{m}(k_j - 1)$, then

$$\sum_{a \in [n]} x_{a,1} \otimes \cdots \otimes x_{a,m} \tag{2}$$

constitutes a unique tensor rank decomposition.

Theorem 1 was proved for $m = 3$ and $\mathbb{F} = \mathbb{R}$ in [3], was later extended to more than three subsystems by Sidiropoulos and Bro [7], and then to an arbitrary field by Rhodes [6] (Landsberg's proof also applies to an arbitrary field [4]).

Our strengthening of Theorem 1 is the following

Theorem 2. *Let $n \geqslant 2$ and $m \geqslant 2$ be integers, $\mathcal{V} = \mathcal{V}_1 \otimes \cdots \otimes \mathcal{V}_m$ be a multipartite vector space. Let E be a multiset of n product tensors* (1). *Assume that*

$$2|S| \leqslant 1 + \sum_{j=1}^{m} (\dim \operatorname{span}\{x_{a,j} : a \in S\} - 1)$$

for all sets $S \subset [n]$ with $|S| \geqslant 2$. Then (2) *constitutes a unique tensor rank decomposition.*

It is not hard to deduce Theorem 1 from Theorem 2. At fact, Theorem 2 implies and generalises many known sufficient conditions of the tensor rank decomposition uniqueness. In turn, Theorem 2 follows from the following

Theorem 3. *Let $n \geqslant 2$ and $m \geqslant 2$ be integers, $\mathcal{V} = \mathcal{V}_1 \otimes \cdots \otimes \mathcal{V}_m$ be a multipartite vector space. Let E be a multiset of n product tensors* (1). *If*

$$\dim \operatorname{span} E \leqslant \sum_{j=1}^{m} (\dim \operatorname{span}\{x_{a,j} : a \in [n]\} - 1)\,,$$

then E splits.

Theorem 2 yields many known results in the tensor rank decomposition area.

In particular, it yields a bound on the number of subsystems $j \in [m]$ for which a circuit of product tensors can have rank greater than one. Our bound improves recent results in this vein [1, 2], and is sharp.

Corollary. *Let n and m be positive integers, and let $\mathcal{V} = \mathcal{V}_1 \otimes \cdots \otimes \mathcal{V}_m$ be a multipartite vector space over a field $\mathbb{F}$. If a set of product tensors* (1) *forms a circuit, then* $\dim \operatorname{span}\{x_{a,j} : a \in [n]\} > 1$ *for at most $n - 2$ indices $j \in [m]$.*

Other applications are listed in [5].

References

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