The integrability condition for the Liénard-like equation

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Abstract. The purpose of this report is the demonstration of searching the first integral of motion by the method of normal form. For the object of this demonstration, we have chosen the Liénard-like equation. We represented the equation as a dynamical system and parameterized it. We wrote down the conditions of the local integrability near stationary points and found parameter values at which these conditions are satisfied at all stationary points simultaneously. The system is integrable at solutions of these conditions.

Introduction

We use the approach based on the local analysis. It uses the resonance normal form calculated near stationary points [1]. In the paper [2] it was suggested the method for searching the values of parameters at which the dynamical system is locally integrable in all stationary points simultaneously. Satisfying such local integrability conditions is a necessary condition of global integrability. For the global integrability of an autonomous planar system, it is enough to have one global integral of motion. From its expression, you can get the solution of the system in quadratures. That is the integrability always leads to solvability. Note, that the converse is not true.

Problem

We will check our method on the example of the Liénard-like equation

$$\ddot{x} = f(x)\dot{x} + g(x), \tag{1}$$

We suppose $f(x)$ is a quadratic polynomial and $g(x)$ is polynomial of fourth-order. Usually in the Liénard equation it supposed $f(x)$ is an odd function. Opposite, we

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suppose $g(x)$ is an voluntary function [3] and say about the Liénard-like equation. Equation (1) is equivalent to the dynamical system

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= (a_0 + a_2 x^2)\, y + d_0(1 + d_1 x)(1 + d_2 x)(1 + d_3 x)(1 + d_4 x), \end{aligned} \tag{2}$$

here x and y are functions in time and parameters $a_0, a_2, d_0, d_1, d_2, d_3, d_4$ are real. If exclude trivial case $d_0 = 0$ it is possible to put $d_0 = 1$.

The problem is to construct the integral of motion of system (2).

Method

The main idea of the discussed method is a search of conditions on the system parameters when the system is locally integrable near its stationary points. The local integrability means we have enough number (one for an autonomic plane system) of the local integrals near each stationary point. Local integrals can be different for each such point, but for the existence of the global integral, the local integrals should exist in all stationary points. This is a necessary condition. We have an algebraic condition for local integrability. It is the condition **A** [1, 2]. We look for sets of parameters at which the condition **A** is satisfied at all stationary points simultaneously. Such sets of parameters are good candidates for the existence of global integrals. These integrals we look for by other methods.

The symmetric notation of the g polynomial in (2) allows to get the condition **A** for all stationary points $(x = -1/d_1, y = 0), \ldots, (x = -1/d_4, y = 0)$ from the normal form for the single point by a permutation of d-parameters. This procedure eliminates the solutions which correspond to a single point only.

Conditions of the Integrability

The condition **A** is some infinite sequence of polynomial equations in coefficients of the system. Near each of the stationary points are its own equations. The normal form has a nontrivial form in the resonance case only. The eigenvalues of the linear part of system (2) are (at $d_0 = 1$)

$$\frac{1}{2}\left(a_0 \mp \sqrt{a_0^2 + 4(d_1 + d_2 + d_3 + d_4)}\right),$$

so we can choose the parameter a_0 to have the resonance $(1 : N)$ by solving the equation

$$\begin{aligned} &\tfrac{1}{2}\left(a_0 - \sqrt{a_0^2 + 4(d_1 + d_2 + d_3 + d_4)}\right) = \\ &-\tfrac{N}{2}\left(a_0 + \sqrt{a_0^2 + 4(d_1 + d_2 + d_3 + d_4)}\right). \end{aligned}$$

We calculated the lowest contributions in conditions A for resonances (1:1), (1:2) and (1:3) and added the ones with permutations of d parameters. We got very complicated algebraic equations in parametric space. Now we are searching for rational solutions.

Result

The case with resonance (1:3) has been explored in [4] by the Puiseux series method. There is the nontrivial first integral at rational numerical values of parameters. These parameter values satisfy the received here equations.

For resonance (1:1) we received at least one rational solution of the truncated condition A. It is

$$a_2 = -a_0 d_1^2, d_0 = 1, d_2 = -d_1, d_3 = 0, d_4 = 0.$$

System (2) at these parameters has a form

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= (1 + a_0 y)\left(1 - d_1^2 x^2\right) \end{aligned}$$

and the corresponded integral of motion is

$$I(x, y) = \frac{a_0 y - \log(a_0 y + 1)}{a_0^2} + \frac{1}{3} d_1^2 x^3 - x.$$

References

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