The integrability condition for the Liénard-like equation

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Integrability

For an ODE system the integrals of motion satisfy the relation

$$\frac{d\ I_k(x_1,\ldots,x_n,t)}{d\ t} = 0 \quad \text{along the system} \quad \frac{d\ x_i}{d\ t} = \phi_i(x_1,\ldots,x_n,t),$$

$$i = 1,\ldots,n, \quad k = 1,\ldots,m$$

The integrals should satisfy also some additional properties.
System is called integrable if it has enough numbers of the integrals.
For integrability of an autonomous plane system, it is enough to have a single integral.

- Integrability is a very important property of the system. In particular, if a system is integrable then it is solvable by quadrature (but not vise versa).
- The knowledge of the integrals is important at the investigation of a phase portrait, for the creation of symplectic integration schemes e.t.c.

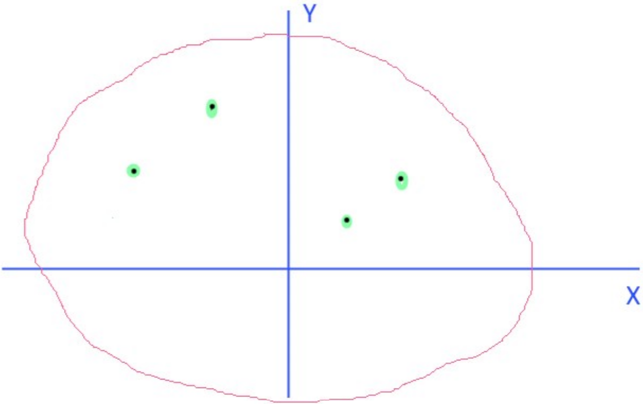
Task

- Generally, integrability is a rare property.
- But the system may depend on parameters.
- The task is to find the values of these parameters at which the system is integrable.

Global & Local

- Let us see the plane autonomous system in the variables $\{x(t), y(t)\}$.
- We look for the global integral a function whose a full derivation in time equal zero along with the system in some domain in the variables $\{x, y\}$.
- If we assume that this function is a single-valued function, then it must satisfy this condition at every point of the integrability domain.
- Thus, the condition local of integrability at every point will be a necessary condition for global integrability.

$$I(X(t), Y(t), a, b, c, \ldots) = \mathrm{const}(t)$$



Idea of the Technique

- The local analysis of ODEs is well developed.
- Generally, the global integral does not exist. But it may exist at some values of the system parameters.
- We solve the necessary conditions of the local integrability in the parametric space. We get sets of parameter values that are good candidates for a search of global integrability.
- At these values of parameters, we will search the global integrals later.

Example

For example, we have had treated the system

$$\begin{aligned} dx/dt &= -y^3 - b\,x^3 y + a_0\,x^5 + a_1\,x^2 y^2, \\ dy/dt &= (1/b)\,x^2 y^2 + x^5 + b_0\,x^4 y + b_1\,x\,y^3, \end{aligned}$$

with five arbitrary real parameters $b \neq 0, a_1, a_2, b_1, b_2$.

Using the Power Geometry method, we brought the system to a non-degenerate form.

With our technique, we found seven two-dimensional conditions at which the system above is integrable

$$
\begin{array}{ll}
1)\ b_1 = 0, & a_0 = 0, a_1 = -b_0\, b,\ \ b^2 \neq 2/3; \\
2)\ b_1 = -2\, a_1, & a_0 = a_1\, b, b_0 = b_1\, b,\ b^2 \neq 2/3; \\
3)\ b_1 = 3/2\, a_1, & a_0 = a_1\, b, b_0 = b_1\, b,\ b^2 \neq 2/3; \\
4)\ b_1 = 8/3\, a_1, & a_0 = a_1\, b, b_0 = b_1\, b,\ b^2 \neq 2/3; \\
5)\ b_1 = 3/2\, a_1, & a_0 = (2b_0 + b\,(3a_1 - 2b_1))/3,\ b = \sqrt{2/3}; \\
6)\ b_1 = 6\, a_1 + 2\sqrt{6} b_0, & a_0 = (2b_0 + b\,(3a_1 - 2b_1))/3,\ b = \sqrt{2/3}; \\
7)\ b_1 = -2/3\, a_1, & 20a_0 + 2\sqrt{6}a_1 + 4b_0 + 3\sqrt{6}b_1 = 0, \\
 & 3a_0 - 2b_0 \neq b\,(3a_1 - 2b_1), \qquad b = \sqrt{2/3}.
\end{array}
$$

For each of these conditions, we found the first integral of motion.

- In this example, we first found those 7 relations which were perspective for integrability, and then the integrals were found by some other methods.
- This example is sophisticate and for a demonstration of our technique, we need to consider a simpler sample.

Condition of the Local Integrability

- The resonance normal form was introduced by Poincaré for the investigation of systems of nonlinear ordinary differential equations. It is based on the maximal simplification of the right-hand sides of these equations by invertible transformations.
- The normal form approach was developed in works of G.D. Birkhoff, T.M. Cherry, A. Deprit, F.G. Gustavson, C.L. Siegel, J. Moser, A.D. Bruno et.al. This technique is based on the Local Analysis method by Prof. Bruno.

Multi-index notation

Let's suppose that we treat the polynomial system and rewrite this n-dimension system in the terms

$$\dot{x}_i = \lambda_i x_i + x_i \sum_{\mathbf{q} \in \mathbb{N}_i} f_{i,\mathbf{q}} \mathbf{y}^{\mathbf{q}}, \quad i = 1, \dots, n, \tag{1}$$

where we use the multi-index notation

$$\mathbf{x}^{\mathbf{q}} \equiv \prod_{j=1}^{n} x_j{}^{q_j},$$

with the power exponent vector $\mathbf{q} = (q_1, \dots, q_n)$
Here the sets:

$$\mathbb{N}_i = \{\mathbf{q} \in \mathbb{Z}^n : q_i \geq -1 \text{ and } q_j \geq 0\,, \text{ if } j \neq i\,, \quad j = 1, \dots, n\}\,,$$

because the factor y_i has been moved out of the sum in (1).

Normal form

The normalization is done with a near-identity transformation:

$$x_i \;=\; z_i + z_i \sum_{\mathbf{q}\in\mathbb{N}_i} h_{i,\mathbf{q}}\mathbf{z}^{\mathbf{q}}, \quad i = 1,\ldots,n \tag{2}$$

after which we have system (1) in the normal form:

$$\dot{z}_i = \lambda_i z_i + z_i \sum_{\substack{\langle \mathbf{q},\mathbf{L}\rangle = 0 \\ \mathbf{q}\in\mathbb{N}_i}} g_{i,\mathbf{q}}\mathbf{z}^{\mathbf{q}}, \quad i = 1,\ldots,n\,. \tag{3}$$

Resonance terms

- The important difference between (1) and (3) is a restriction on the range of the summation, which is defined by the equation:

$$\langle \mathbf{q}, \mathbf{L} \rangle = \sum_{j=1}^{n} q_j \lambda_j = 0 . \tag{4}$$

- **L** here is the vector of the eigenvalues of matrix of linear part of the system (1). The **q**-terms in the normal form (3) are terms, for which (4) is valid. They are called resonance terms.

Note, if the eigenvalues are not comparable then condition (4) is never valid at any components of the vector **q**, because they are integer. For example such situation takes place if $\lambda_1 = 1, \lambda_2 = \sqrt{2}$. In that case the normal form (3) will be a linear system.

Calculation of the Normal form

The h and g coefficients in (2) and (3) are found by using the recurrence formula:

$$g_{i,\mathbf{q}} + \langle \mathbf{q}, \mathbf{L} \rangle \cdot h_{i,\mathbf{q}} = -\sum_{j=1}^{n} \sum_{\substack{\mathbf{p}+\mathbf{r}=\mathbf{q} \\ \mathbf{p},\mathbf{r} \in \bigcup_i \mathbb{N}_i \\ \mathbf{q} \in \mathbb{N}_i}} (p_j + \delta_{ij}) \cdot h_{i,\mathbf{p}} \cdot g_{j,\mathbf{r}} + \tilde{\Phi}_{i,\mathbf{q}} \,, \quad (5)$$

For this calculation, we wrote two programs, in LISP and the high-level language of the MATHEMATICA system.

Conditions A and ω

There are two conditions:

- Condition A. In the normal form (3)

$$g_j(Z) = \lambda_j a(Z) + \bar{\lambda}_j b(Z), \quad j = 1, \dots, n,$$

  where $a(Z)$ and $b(Z)$ are some formal power series.
- Condition ω (on small divisors). It is fulfilled for almost all vectors **L**. At least it is satisfied at rational eigenvalues.
- If these conditions are satisfied then the normalizing transformation (2) converges.

Near a stationary point the condition **A:**

- Ensures convergence.
- Provides the local integrability.
- Isolates the periodic orbits if eigenvalues are pure imagine.

Near a stationary point the condition **A**

- The condition **A is an infinite system of algebraic equations on the system parameters.**
- **It can be calculated by the CA program till some finite order. It will be the necessary condition of the local integrability as a finite system of algebraic equations in the system parameters. It is solvable.**

The Liénard equation

- Let f and g be two continuously differentiable functions on $\mathbb{R}$, where g is an odd and f is an even function. Then the second-order ordinary differential equation of the form

$$\ddot{x} = f(x)\dot{x} + g(x) = 0$$

  is called the Liénard-like equation.
- **Theorem** Liénard equation has a unique and stable limit cycle surrounding the origin if it satisfies the following additional properties...
- We will look now at the case of the even function f but the arbitrary function g.

Parametrization

- We choose f and g as the polynomials

$$\dot{x} = y,$$
$$\dot{y} = (a_0 + a_2 x^2)\, y + d_0(1 + d_1 x)(1 + d_2 x)(1 + d_3 x)(1 + d_4 x).$$

  x and y are functions in time and parameters $a_0, a_2, d_0, d_1, d_2, d_3, d_4$ are real. If exclude trivial case $d_0 = 0$ it is possible to put $d_0 = 1$.
- There are four (or fewer) stationary points here.
- The symmetric notation of the g polynomial allows to get the condition **A** for all stationary points $(x = -1/d_1, y = 0), \ldots, (x = -1/d_4, y = 0)$ from the condition for the single point just by a permutation of the d-parameters.

Conditions of the Integrability

- The integrability takes place in the resonance case only.
- The eigenvalues of the linear part of our system (for a diagonalizable case) are

$$\frac{1}{2}\left(a_0 \mp \sqrt{a_0^2 + 4d_0(d_1 + d_2 + d_3 + d_4)}\right).$$

- So, we can choose the parameter a_0 to have the resonance $(1:N)$ by solving the square equation

$$\tfrac{1}{2}\left(a_0 - \sqrt{a_0^2 + 4d_0(d_1 + d_2 + d_3 + d_4)}\right) = -\tfrac{N}{2}\left(a_0 + \sqrt{a_0^2 + 4d_0(d_1 + d_2 + d_3 + d_4)}\right).$$

- With the fixed a_0 we calculated the normal form for resonances (1:1), (1:2), and (1:3) separately till the fourth, sixth and eighth order.
- For each case, we got algebraic systems in the parametric space with 2x4 equations. They are huge.

The system for the resonance (1:1)

$$
\begin{aligned}
L1 = \{ & -9\, a2\, d1^4 (d1 - d2)^3 (d1 - d3)^3 (d1 - d4)^3 \left(d1^3 + 5\, d2\, d3\, d4 + d1^2 (d2 + d3 + d4) - 3\, d1 (d3\, d4 + d2 (d3 + d4))\right) == 0, \\
& a2\, d1^6 \left(-20\, a2^2 (d1 - d2)(d1 - d3)(d1 - d4)\left(d1^3 + 5\, d2\, d3\, d4 + d1^2 (d2 + d3 + d4) - 3\, d1 (d3\, d4 + d2 (d3 + d4))\right) + \right. \\
& \quad d1^2 \left(261\, d2^3 d3^3 d4^3 - 471\, d1\, d2^2 d3^2 d4^2 (d3\, d4 + d2 (d3 + d4)) + \right. \\
& \quad\quad d1^7 \left(20\, d2^2 + 20\, d3^2 + 31\, d3\, d4 + 20\, d4^2 + 31\, d2 (d3 + d4)\right) + \\
& \quad\quad d1^2\, d2\, d3\, d4 \left(343\, d3^2 d4^2 + 656\, d2\, d3\, d4 (d3 + d4) + d2^2 \left(343\, d3^2 + 656\, d3\, d4 + 343\, d4^2\right)\right) + \\
& \quad\quad d1^5 \left(-71\, d2^3 (d3 + d4) + d3\, d4 \left(-71\, d3^2 + 76\, d3\, d4 - 71\, d4^2\right) + d2^2 \left(76\, d3^2 + 309\, d3\, d4 + 76\, d4^2\right) + \right. \\
& \quad\quad\quad \left. d2 \left(-71\, d3^3 + 309\, d3^2 d4 + 309\, d3\, d4^2 - 71\, d4^3\right)\right) + \\
& \quad\quad d1^6 \left(20\, d2^3 - 80\, d2^2 (d3 + d4) - d2 \left(80\, d3^2 + 237\, d3\, d4 + 80\, d4^2\right) + 20 \left(d3^3 - 4\, d3^2 d4 - 4\, d3\, d4^2 + d4^3\right)\right) + \\
& \quad\quad d1^4 \left(104\, d3^2 d4^2 (d3 + d4) + d2^3 \left(104\, d3^2 + 257\, d3\, d4 + 104\, d4^2\right) + d2\, d3\, d4 \left(257\, d3^2 - 180\, d3\, d4 + 257\, d4^2\right) + \right. \\
& \quad\quad\quad \left. 4\, d2^2 \left(26\, d3^3 - 45\, d3^2 d4 - 45\, d3\, d4^2 + 26\, d4^3\right)\right) - \\
& \quad\quad d1^3 \left(93\, d3^3 d4^3 + 409\, d2\, d3^2 d4^2 (d3 + d4) + d2^2 d3\, d4 \left(409\, d3^2 + 192\, d3\, d4 + 409\, d4^2\right) + \right. \\
& \quad\quad\quad \left.\left.\left. d2^3 \left(93\, d3^3 + 409\, d3^2 d4 + 409\, d3\, d4^2 + 93\, d4^3\right)\right)\right)\right) == 0 \};
\end{aligned}
$$

The solution for the resonance (1:1)

```
L = L1 /. d1 → d;
L2 = (L /. d2 → d1) /. d → d2;
L3 = (L /. d3 → d1) /. d → d3;
L4 = (L /. d4 → d1) /. d → d4;
```

```
Solve[Join[L1, L2, L3, L4]]
```

решит… соединить

```
{{a2 → 0}, {d1 → 0, d2 → 0, d3 → -d4}, {d1 → 0, d2 → -d4, d3 → 0}, {d1 → 0, d3 → -d2, d4 → 0}, {d1 → 0, d2 → 0, d3 → -d4},
 {d1 → 0, d3 → 0, d4 → -d2}, {d1 → -d2, d3 → 0, d4 → 0}, {d1 → -d2, d3 → 0, d4 → 0}, {d1 → d2, d3 → d2, d4 → 0}, {d1 → d2, d3 → d2, d4 → d2},
 {d1 → -d3, d2 → 0, d4 → 0}, {d1 → d3, d2 → d3, d4 → 0}, {d1 → d3, d2 → d3, d4 → d3}, {d1 → -d4, d2 → 0, d3 → 0}, {d1 → -d4, d2 → 0, d3 → 0},
 {d1 → d4, d2 → d4, d3 → 0}, {d1 → d4, d2 → d4, d3 → d4}, {d1 → d4, d2 → 0, d3 → d4}, {d2 → 0, d3 → -d1, d4 → 0}, {d2 → 0, d3 → 0, d4 → -d1},
 {d2 → -d1, d3 → 0, d4 → 0}, {d2 → d1, d3 → d1, d4 → 0}, {d2 → d1, d3 → 0, d4 → d1}, {d2 → d1, d3 → d1, d4 → d1}, {d1 → 0, d2 → d3, d4 → d3},
 {d1 → d3, d2 → d3, d4 → d3}, {d1 → 0, d2 → -d4, d3 → 0}, {d1 → 0, d2 → -d4, d3 → 0}, {d1 → d4, d2 → d4, d3 → d4},
 {d1 → 0, d2 → d4, d3 → d4}, {d1 → 0, d2 → -d4, d3 → 0}, {d2 → 0, d3 → -d1, d4 → 0}, {d2 → 0, d3 → -d1, d4 → 0}, {d2 → d1, d3 → d1, d4 → d1},
 {d1 → 0, d3 → -d2, d4 → 0}, {d1 → 0, d3 → -d2, d4 → 0}, {d1 → 0, d3 → d2, d4 → d2}, {d1 → d2, d3 → d2, d4 → d2}, {d1 → 0, d2 → 0, d3 → -d4},
 {d1 → 0, d2 → 0, d3 → -d4}, {d1 → d4, d2 → d4, d3 → d4}, {d1 → 0, d2 → d4, d3 → d4}, {d1 → 0, d3 → -d2, d4 → 0},
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 {d2 → d1, d3 → d1, d4 → d1}, {d2 → 0, d3 → d1, d4 → d1}, {d1 → d2, d3 → 0, d4 → d2}, {d1 → d2, d3 → d2, d4 → d2},
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 {d1 → 0, d2 → 0, d3 → 0, d4 → 0}, {d1 → 0, d2 → 0, d3 → 0, d4 → 0}, {d1 → 0, d2 → 0, d3 → 0, d4 → 0}, {d1 → 0, d2 → 0, d3 → 0, d4 → 0}}
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First Integrals of Motion

For searching for global integrals, we use the Lagutinsky method. We divided the left and right sides of the system equations into each other.

$$\frac{dx(t)}{dt} = P(x(t), y(t)),$$

$$\frac{dy(t)}{dt} = Q(x(t), y(t)).$$

In result we have the first-order differential equations for $x(y)$ or $y(x)$

$$\frac{dx}{dy} = \frac{P(x,y)}{Q(x,y)} \quad \text{or} \quad \frac{dy}{dx} = \frac{Q(x,y)}{P(x,y)}.$$

Then we solved them by the MATHEMATICA solver and got cumbersome solutions $y(x)$. After that we calculated the integrals from these solutions by extracting the integration constants $I(x(t), y(t)) = const(t)$.

- For the resonance (1:1) we received at least one rational solution of the truncated condition A. It is

$$a_2 = -a_0 d_1^2, d_0 = 1, d_2 = -d_1, d_3 = 0, d_4 = 0.$$

- The system at these parameters has a form

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= (1 + a_0 y)\left(1 - d_1^2 x^2\right) \end{aligned}$$

- The first integral of motion is

$$I(t) = \frac{a_0 y - \log(a_0 y + 1)}{a_0^2} + \frac{1}{3} d_1^2 x^3 - x.$$

  Its time derivative $dI(t)/dt = 0$ along with the system's overall phase space. So, it is the first integral.
- Professor N.V. Demina kindly provided me the partial result for the case (1:3). The values of the system parameters satisfy the corresponded equation.

Note

- In the original terms the equation above is

$$\ddot{x} + a_0(d_1^2 x^2 - 1)\dot{x} + (d_1^2 x^2 - 1) = 0$$

- and van der Pol's equation is

$$\ddot{x} + a_0(d_1^2 x^2 - 1)\dot{x} + x = 0$$

Conclusion

- We represented the Liénard equation as a dynamical system and parameterized it in a polynomial form. For this system, we found at least one set of parameters at which it has the first integrals of motion and, in this way solvable.
- The workability of the described pure algebraic technique was illustrated.
- The proposed method for finding suitable integrability parameters has no restrictions on the dimension of the system. It is suitable for any autonomous polynomial system which is solved concerning the derivations.

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