Decomposition of a Quantum System Into Subsystems in Finite Quantum Mechanics

Polynomial Computer Algebra '2021
April 19-24, 2021
Euler International Mathematical Institute, St. Petersburg, Russia

Vladimir Kornyak

Laboratory of Information Technologies
Joint Institute for Nuclear Research
Dubna, Russia

April 19, 2021
Quantum mereology

- Mereology
  - relations of part to whole
  - relations of part to part within a whole

- Quantum mereology
  whole = isolated quantum system (= Universe)
  - bipartite decomposition of quantum system into system and environment
  - interaction of system and observer
  - emergence of space and time from quantum entanglement
  - ...

Tasks:
study of decomposition methods,
calculation of quantitative separation and correlation characteristics:
quantum correlations and interaction energies between subsystems,
metrics for emergent geometry
Factorization of Hilbert space I

Notation

- $\mathcal{H}$ is global Hilbert space
- $\mathcal{H}_1, \ldots, \mathcal{H}_K$ are local Hilbert spaces, $d_k = \dim \mathcal{H}_k$
- Integer decomposition $\dim \mathcal{H} = d_1 d_2 \cdots d_K$

- $|i\rangle$ is $i$th element of orthonormal basis
  - $|0\rangle = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$, $|1\rangle = \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix}$, $|d-1\rangle = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}$

in $d$-dimensional Hilbert space
Factorization of Hilbert space II

**Direct** procedure: given $\mathcal{H}_1, \ldots, \mathcal{H}_K \rightarrow \mathcal{H}$

orthonormal basis in $\mathcal{H}$:

$$|i\rangle = |i_1\rangle \otimes \cdots \otimes |i_k\rangle \otimes \cdots \otimes |i_K\rangle$$

$$i = i_1 (d_2 d_3 \cdots d_K) + \ldots + i_k (d_{k+1} \cdots d_K) + \ldots + i_{K-1} (d_K) + i_k$$

**Reverse** procedure: given $(\mathcal{H}, d_1 d_2 \cdots d_K = \dim \mathcal{H}) \rightarrow \mathcal{H}_1, \ldots, \mathcal{H}_K$

fixing unitary freedom in bases

$$U |\psi\rangle = U_1 |\psi_1\rangle \otimes \cdots \otimes U_K |\psi_K\rangle = (U_1 \otimes \cdots \otimes U_K) (|\psi_1\rangle \otimes \cdots \otimes |\psi_K\rangle)$$

$$\Rightarrow \left( U_1 \otimes \cdots \otimes U_K \right)^{-1} U |\psi\rangle = |\psi_1\rangle \otimes \cdots \otimes |\psi_K\rangle$$

Factorization of $\mathcal{H}$ is specified by

dimension decomposition $\dim \mathcal{H} = d_1 d_2 \cdots d_K$ and global unitary transformation $U$
Finite QM: Permutation Hilbert space $\mathcal{H}_N$

- **Basis:** $\Omega = \{e_1, \ldots, e_N\} \cong \{1, \ldots, N\}$, primary ("ontic") objects
- **Symmetry group on $\Omega$:** $G \leq S_N$, best choice for physics $G = S_N$
  - Exponent of $G$: $\ell = \text{lcm}_{g \in G} \text{order}(g)$
- **Field:** $\mathcal{F} = \mathbb{Q}\left(e^{2\pi i/\ell}\right)$, a dense subfield of QM field $\mathbb{C}$ for $\ell > 2$
- **Permutation representation in $\mathcal{H}_N$:** $\mathcal{P}(g)_{i,j} = \delta_{ig,j}$
Finite QM: Decomposition of permutation representation

- Trivial subspace $\mathcal{H}_{\text{triv}} = \text{span}\left( |\omega\rangle = \left(1, 1, \ldots, 1\right)^\top_N \right)$

- Standard space $\mathcal{H}_* = \mathcal{H}_{\text{triv}}^\perp = P_\star\mathcal{H}_N$, \quad $P_\star = 1_N - \frac{|\omega\rangle\langle\omega|}{N}$

Quantum mechanics manifests itself precisely in $\mathcal{H}_*$

Tom Banks
in “Finite Deformations of Quantum Mechanics” arXiv:2001.07662:

“The work . . . shows that . . . (the projection on the $S_N$ singlet subspace of the Hilbert space), naturally generates a set of truly quantum systems, which can encompass finite dimensional approximations to all known models of theoretical physics.”

Banks argues (very convincing) that true choice must be $G = S_N$, where $N$ is the number of fundamental (Planck) elements.

$N$ is estimated as $\sim \text{Exp(Exp(20))}$ for 1 cm$^3$ of matter
and $\sim \text{Exp(Exp(123))}$ for the entire Universe.
Finite QM: Quantum states as projections of natural vectors

Projection $\mathcal{H}_N \xrightarrow{P_*} \mathcal{H}_* \text{ of natural vectors}$

$|n\rangle = (n_1, \ldots, n_N)^\top \in \mathbb{N}^N \subset \mathcal{H}_N$

reproduces all quantum states in standard space $\mathcal{H}_*$

for rational-representation group $S_N$

$m^{th}$ order natural vectors: coordinates $n_i \in \{0, 1, \ldots, m\}, m \geq 2$

No. of quantum states that stem from $m^{th}$ order natural vectors $\geq 2^N - 2$ — exponential growth

Area of unit sphere in nonnegative orthant $S_{+}^{N-1}$

is $\frac{\pi^{N/2}}{2^N \Gamma(N/2 + 1)} \approx \sqrt{\frac{N}{\pi}} \left(\frac{e\pi}{2N}\right)^{N/2}$ — more than exponential decrease

$\implies m^{th}$ order vectors represent a significant part of quantum states
Ontic vectors are 2\textsuperscript{nd} order natural vectors

- ontic vector $|q\rangle$ is a **bit string** of length $N$
  or, equivalently, a **partition** of ontic set into two nontrivial subsets

$$\Omega = \Omega_q \bigcup \Omega_{\sim q}, \quad \Omega_{\sim q} = \Omega \setminus \Omega_q$$

- normalized **inner product** in standard space $\mathcal{H}_+$

$$S(q, r) = \frac{N\langle q \& r \rangle - \langle q \rangle \langle r \rangle}{\sqrt{\langle q \rangle \langle \sim q \rangle \langle r \rangle \langle \sim r \rangle}}$$

$\langle \cdot \rangle$ is the **Hamming weight** or population number $= \text{number of 1's}$

- **symmetry** between subset $\Omega_a$ and its complement $\Omega_{\sim a}$ together with obvious identities

$$\langle \sim a \rangle = N - \langle a \rangle \quad \text{and} \quad \langle a \& b \rangle + \langle a \& \sim b \rangle = \langle a \rangle$$

imply

$$S(q, r) = -S(\sim q, r) = -S(q, \sim r) = S(\sim q, \sim r)$$
Ontic density matrix in ontic basis

- **ontic basis**: original permutation basis in $\mathcal{H}_N$, i.e., the set $\Omega$
- **ontic density matrix**: density matrix in $\mathcal{H}_\star$ for ontic state $|q\rangle \in \mathcal{H}_N$

$$\rho^o_q = \frac{P_\star |q\rangle \langle q| P_\star}{\langle q| P_\star |q\rangle} = \frac{1}{N} \frac{(|q\rangle - \alpha |\omega\rangle)(\langle q| - \alpha \langle \omega|)}{\alpha (1 - \alpha)}$$

$$\alpha = \frac{\langle q\rangle}{N}$$ is the population density

- **duality**

$$\rho^o_q \xrightarrow{\alpha \to 1 - \alpha} \rho^o_{\sim q}$$
Energy basis I

Evolution of isolated system

- **in continuous QM**: one-parameter unitary group \( U_t = e^{-iHt} \)
generated by Hamiltonian \( H \) whose *eigenvalues* are called *energies*

- **in finite QM**: representation of cyclic group \( U(g)^t \) generated by an element \( g \in G \)

  We call energy basis an orthonormal basis in which 
  matrix \( P(g) \) in \( H_N \) is diagonal

- Any permutation is a product of disjoint cycles

- **Total number of \( \ell \)-cycles**
  in whole \( S_N \) is
  \[
  \frac{N!}{\ell} = \frac{N!}{\ell! (N-\ell)!} \times (\ell-1)! \times (N-\ell)!
  \]

  or expected number in a single permutation \( \frac{1}{\ell} \)

  \( \implies \) high-energy evolutions are more common

  high-energy = high-frequency by Plank formula \( E = h\nu \)
Energy basis II

- $\ell$-cycle matrix in ontic basis:

$$C_\ell = \begin{pmatrix}
0 & 1 & 0 & \cdots & 0 \\
0 & 0 & 1 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
1 & 0 & 0 & \cdots & 0 \\
\end{pmatrix}$$

- $\ell$-cycle matrix in energy basis:

$$F_\ell C_\ell F_\ell^{-1} = \begin{pmatrix}
1 & 0 & 0 & \cdots & 0 \\
0 & \zeta_\ell & 0 & \cdots & 0 \\
0 & 0 & \zeta_\ell^2 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & \zeta_\ell^{\ell-1} \\
\end{pmatrix}$$

$$\zeta_\ell = e^{2\pi i/\ell}$$ is the $\ell$th primitive root of unity.
Energy basis III

- Fourier transform:

\[ F_{\ell} = \frac{1}{\sqrt{\ell}} \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & \zeta_{\ell}^{-1} & \zeta_{\ell}^{-2} & \cdots & \zeta_{\ell}^{-(\ell-1)} \\ 1 & \zeta_{\ell}^{-2} & \zeta_{\ell}^{-4} & \cdots & \zeta_{\ell}^{-(2\ell-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \zeta_{\ell}^{-(\ell-1)} & \zeta_{\ell}^{-(2\ell-1)} & \cdots & \zeta_{\ell}^{-(\ell-1)(\ell-1)} \end{pmatrix} \]

- Matrix of transition from ontic to energy basis

\[ \mathcal{P}(g) = \bigoplus_{m=1}^{M} C_{\ell m} \quad \text{is} \quad F = \bigoplus_{m=1}^{M} F_{\ell m} \]

Ontic density matrix in energy basis

\[ \rho^{e}_{q} = F \rho^{o}_{q} F^{-1} \]
Entanglement measures

Common types of entropies underlying quantum correlation measures

- Rényi family $S_\alpha(\rho) = \frac{1}{1-\alpha} \log \text{tr}(\rho^\alpha)$, $\alpha \geq 0$, $\alpha \neq 1$
- von Neumann entropy $S_1(\rho) = -\text{tr}(\rho \log \rho)$, most common in physics

We prefer 2nd Rényi entropy (collision entropy) $S_2(\rho) = -\log \text{tr}(\rho^2)$:

- easy to calculate: $\text{tr}(\rho^2) = \sum_{i=1}^{n} \rho_{ii}^2 + 2 \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} |\rho_{ij}|^2$
- $\text{tr}(\rho^2)$ is Born’s probability: “the system observes itself”
- $\text{tr}(\rho^2)$ is Frobenius inner square of $\rho$
  Frobenius (Hilbert-Schmidt) inner product is the most natural metric on matrices
Illustrative calculation: bipartite decompositions of homogeneous system

### Subsystem size vs. Entropy

| Subsystem size $|A|$ | $S_2(\rho_A)$ |
|-----------------|---------------|
| 0               | 0             |
| 1               | $\approx 2032$ |
| 2               | $\approx 2037$ |
| 3               | $\approx 2037$ |
| 4               | $\approx 2029$ |
| 5               | $\approx 2050$ |
| 6               | $\approx 2053$ |
| 7               | $\approx 2058$ |
| 8               | $\approx 2112$ |
| 9               | $\approx 2035$ |
| 10              | $\approx 2040$ |
| 11              | $\approx 2029$ |

#### Graphical Representation

- **Weak dependence on quantum state**: visually, all graphs are almost identical: behavior for **sufficiently large** number of decomposition components $|X|$
- **Symmetry** $S_2(\rho_A) = S_2(\rho_{X\setminus A})$ is a manifestation of **Schmidt decomposition** of a pure state: $\rho_A$ and $\rho_{X\setminus A}$ have identical sets of nonzero eigenvalues
- For $|A| < |X|/2$, $\rho_A$ is close to the **maximally mixed state**: $S_2(\rho_A) \approx |A| \log d$