Decomposition of a Quantum System Into Subsystems in Finite Quantum Mechanics

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Quantum mereology

- Mereology
  - relations of part to whole
  - relations of part to part within a whole
- Quantum mereology
  whole = isolated quantum system (= Universe)
  - bipartite decomposition of quantum system into system and environment
  - interaction of system and observer
  - emergence of space and time from quantum entanglement
  - ...

Tasks:
study of decomposition methods,
calculation of quantitative separation and correlation characteristics:
quantum correlations and interaction energies between subsystems,
metrics for emergent geometry

Factorization of Hilbert space I

Notation

- $\mathcal{H}$ is global Hilbert space
- $\mathcal{H}_1, \ldots, \mathcal{H}_K$ are local Hilbert spaces, $d_k = \dim \mathcal{H}_k$
- Integer decomposition $\dim \mathcal{H} = d_1 d_2 \cdots d_K$
- $|i\rangle$ is ith element of orthonormal basis

$$|0\rangle = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad |1\rangle = \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix}, \ldots, |d-1\rangle = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}$$

in d-dimensional Hilbert space

Factorization of Hilbert space II

Direct procedure: given $\mathcal{H}_1, \ldots, \mathcal{H}_K \longrightarrow \mathcal{H}$

orthonormal basis in $\mathcal{H}$:

$$|i\rangle = |i_1\rangle \bigotimes \cdots \bigotimes |i_k\rangle \bigotimes \cdots \bigotimes |i_K\rangle$$

$$i = i_1 \left(d_2 d_3 \cdots d_K\right) + \ldots + i_k \left(d_{k+1} \cdots d_K\right) + \ldots + i_{K-1} \left(d_K\right) + i_K$$

Reverse procedure: given $\left(\mathcal{H}, d_1 d_2 \cdots d_K = \dim \mathcal{H}\right) \longrightarrow \mathcal{H}_1, \ldots, \mathcal{H}_K$

fixing unitary freedom in bases

$$U|\psi\rangle = U_1|\psi_1\rangle \bigotimes \cdots \bigotimes U_K|\psi_K\rangle = \left(U_1 \bigotimes \cdots \bigotimes U_K\right)\left(|\psi_1\rangle \bigotimes \cdots \bigotimes |\psi_K\rangle\right)$$

$$\Longrightarrow \underbrace{\left(U_1 \bigotimes \cdots \bigotimes U_K\right)^{-1} U}_{\text{new } U}|\psi\rangle = |\psi_1\rangle \bigotimes \cdots \bigotimes |\psi_K\rangle$$

Factorization of $\mathcal{H}$ is specified by

dimension decomposition $\dim \mathcal{H} = d_1 d_2 \cdots d_K$ and global unitary transformation U

Permutation Hilbert space $\mathcal{H}_\mathcal{N}$

- Basis: $\Omega = \{e_1, \dots, e_\mathcal{N}\} \cong \{1, \dots, \mathcal{N}\}$, primary ("ontic") objects
- Symmetry group on Ω: $G \leq S_\mathcal{N}$, best choice for physics $G = S_\mathcal{N}$
  - Exponent of G: $\ell = \operatorname{lcm}_{g \in G} \operatorname{order}(g)$
- Field: $\mathcal{F} = \mathbb{Q}\left(e^{2\pi \mathrm{i}/\ell}\right)$, a dense subfield of QM field $\mathbb{C}$ for $\ell > 2$
- Permutation representation in $\mathcal{H}_\mathcal{N}$: $\mathcal{P}(g)_{i,j} = \delta_{ig,j}$

Finite QM: Decomposition of permutation representation

- Trivial subspace $\mathcal{H}_{\rm triv} = {\rm span}\left(|\omega\rangle = (\underbrace{1, 1, \ldots, 1}_{\mathcal{N}})^\top\right)$
- Standard space $\mathcal{H}_\star = \mathcal{H}_{\rm triv}^\perp = {\rm P}_\star \mathcal{H}_\mathcal{N}, \qquad {\rm P}_\star = \mathbb{1}_\mathcal{N} - \frac{|\omega\rangle\langle\omega|}{\mathcal{N}}$

  Quantum mechanics manifests itself precisely in $\mathcal{H}_\star$

  Tom Banks
  in "Finite Deformations of Quantum Mechanics" arXiv:2001.07662:

  "The work ... shows that ... (the projection on the $S_\mathcal{N}$ singlet subspace of the Hilbert space), naturally generates a set of truly quantum systems, which can encompass finite dimensional approximations to all known models of theoretical physics."

  Banks argues (very convincing) that true choice must be $G = S_\mathcal{N}$, where $\mathcal{N}$ is the number of fundamental (Planck) elements.

  $\mathcal{N}$ is estimated as $\sim {\rm Exp}({\rm Exp}(20))$ for 1 cm^3 of matter and $\sim {\rm Exp}({\rm Exp}(123))$ for the entire Universe.

Finite QM: Quantum states as projections of natural vectors

Projection $\mathcal{H}_{\mathcal{N}} \xrightarrow{\mathrm{P}_\star} \mathcal{H}_\star$ of natural vectors
$|n\rangle = (n_1, \ldots, n_{\mathcal{N}})^\top \in \mathbb{N}^{\mathcal{N}} \subset \mathcal{H}_{\mathcal{N}}$
reproduces all quantum states in standard space $\mathcal{H}_\star$

for rational-representation group $\mathsf{S}_{\mathcal{N}}$

m^{th} order natural vectors: coordinates $n_i \in \{0, 1, \ldots, m\}, m \geq 2$

No. of quantum states that stem
from m^{th} order natural vectors $\geq 2^{\mathcal{N}} - 2$ — exponential growth

Area of unit sphere in nonnegative orthant $\mathbb{S}_+^{\mathcal{N}-1}$
is $\dfrac{\mathcal{N}\pi^{\mathcal{N}/2}}{2^{\mathcal{N}}\Gamma(\mathcal{N}/2+1)} \approx \sqrt{\dfrac{\mathcal{N}}{\pi}}\left(\dfrac{\mathrm{e}\,\pi}{2\mathcal{N}}\right)^{\mathcal{N}/2}$ — more than exponential decrease

$\Longrightarrow$ m^{th} order vectors represent a significant part of quantum states

Ontic vectors are 2^{nd} order natural vectors

- ontic vector $|q\rangle$ is a bit string of length $\mathcal{N}$
  or, equivalently, a partition of ontic set into two nontrivial subsets

$$\Omega = \Omega_q \bigsqcup \Omega_{\sim q}, \quad \Omega_{\sim q} = \Omega \setminus \Omega_q$$

- normalized inner product in standard space $\mathcal{H}_\star$

$$S(q,r) = \frac{\mathcal{N}\langle q \,\&\, r\rangle - \langle q\rangle \langle r\rangle}{\sqrt{\langle q\rangle \langle \sim q\rangle \langle r\rangle \langle \sim r\rangle}}$$

  $\langle \cdot \rangle$ is the Hamming weight or population number = number of 1's

- symmetry between subset Ω_a and its complement $\Omega_{\sim a}$ together with obvious identities

$$\langle \sim a\rangle = \mathcal{N} - \langle a\rangle \quad \text{and} \quad \langle a \,\&\, b\rangle + \langle a \,\&\, \sim b\rangle = \langle a\rangle$$

  imply

$$S(q,r) = -S(\sim q, r) = -S(q, \sim r) = S(\sim q, \sim r)$$

Ontic density matrix in ontic basis

- ontic basis: original permutation basis in $\mathcal{H}_{\mathcal{N}}$, ie, the set Ω
- ontic density matrix: density matrix in $\mathcal{H}_\star$ for ontic state $|q\rangle \in \mathcal{H}_{\mathcal{N}}$

$$\rho^o_q = \frac{\mathrm{P}_\star |q\rangle\langle q| \mathrm{P}_\star}{\langle q |\mathrm{P}_\star| q\rangle} = \frac{1}{\mathcal{N}} \frac{(|q\rangle - \alpha |\omega\rangle)(\langle q| - \alpha \langle\omega|)}{\alpha(1-\alpha)}$$

$\alpha = \frac{\langle q\rangle}{\mathcal{N}}$ is the population density

- duality

$$\rho^o_q \xrightarrow[\alpha \to 1-\alpha]{q \to \sim q} \rho^o_{\sim q}$$

Energy basis I

Evolution of isolated system

- in continuous QM: one-parameter unitary group $U_t = \mathrm{e}^{-\mathrm{i}Ht}$ generated by Hamiltonian H whose eigenvalues are called energies
- in finite QM: representation of cyclic group $U(g)^t$ generated by an element $g \in G$
  We call energy basis an orthonormal basis in which matrix $\mathcal{P}(g)$ in $\mathcal{H}_{\mathcal{N}}$ is diagonal

- Any permutation is a product of disjoint cycles
- Total number of $\boldsymbol{\ell}$-cycles
  in whole $\mathsf{S}_{\mathcal{N}}$ is $\dfrac{\mathcal{N}!}{\boldsymbol{\ell}} = \dfrac{\mathcal{N}!}{\ell!\,(\mathcal{N}-\ell)!} \times (\ell-1)! \times (\mathcal{N}-\ell)!$
  or expected number in a single permutation $= \dfrac{1}{\boldsymbol{\ell}}$
  $\Longrightarrow$ high-energy evolutions are more common

high-energy = high-frequency by Plank formula $E = h\nu$

Energy basis II

- ℓ-cycle matrix in ontic basis:

$$C_\ell = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & 0 & 0 & \cdots & 0 \end{pmatrix}$$

- ℓ-cycle matrix in energy basis:

$$F_\ell C_\ell F_\ell^{-1} = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & \zeta_\ell & 0 & \cdots & 0 \\ 0 & 0 & \zeta_\ell^2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \zeta_\ell^{\ell-1} \end{pmatrix}$$

$\zeta_\ell = \mathrm{e}^{2\pi \mathbf{i}/\ell}$ is ℓth primitive root of unity

Energy basis III

- Fourier transform:

$$F_\ell = \frac{1}{\sqrt{\ell}} \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & \zeta_\ell^{-1} & \zeta_\ell^{-2} & \cdots & \zeta_\ell^{-(\ell-1)} \\ 1 & \zeta_\ell^{-2} & \zeta_\ell^{-4} & \cdots & \zeta_\ell^{-2(\ell-1)} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 1 & \zeta_\ell^{-(\ell-1)} & \zeta_\ell^{-2(\ell-1)} & \cdots & \zeta_\ell^{-(\ell-1)(\ell-1)} \end{pmatrix}$$

- Matrix of transition from ontic to energy basis diagonalizing permutation $\mathcal{P}(g) = \bigoplus_{m=1}^{M} C_{\ell_m}$ is $F = \bigoplus_{m=1}^{M} F_{\ell_m}$

Ontic density matrix in energy basis

$$\rho_q^e = F \rho_q^o F^{-1}$$

Entanglement measures

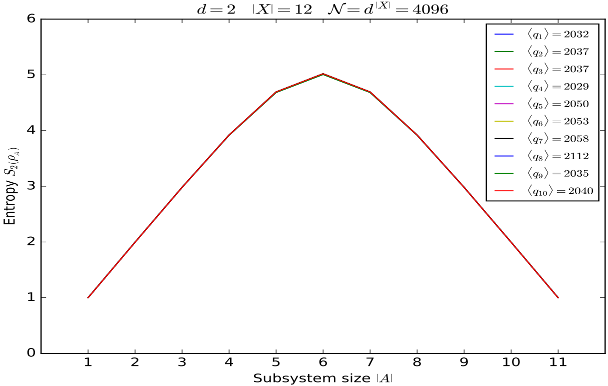
Common types of entropies underlying quantum correlation measures

- Rényi family $S_\alpha(\rho) = \frac{1}{1-\alpha} \log \mathrm{tr}(\rho^\alpha)\,, \quad \alpha \geq 0,\ \alpha \neq 1$
- von Neumann entropy $S_1(\rho) = -\,\mathrm{tr}(\rho \log \rho)$, most common in physics

We prefer 2nd Rényi entropy (collision entropy) $S_2(\rho) = -\log \mathrm{tr}\left(\rho^2\right)$:

- easy to calculate: $\mathrm{tr}\left(\rho^2\right) = \sum_{i=1}^{n} \rho_{ii}^2 + 2 \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} |\rho_{ij}|^2$
- $\mathrm{tr}\left(\rho^2\right)$ is Born's probability: "the system observes itself"
- $\mathrm{tr}\left(\rho^2\right)$ is Frobenius inner square of ρ
  Frobenius (Hilbert-Schmidt) inner product is the most natural metric on matrices

Illustrative calculation: bipartite decompositions of homogeneous system



- Weak dependence on quantum state: visually, all graphs are almost identical: behavior for sufficiently large number of decomposition components $|X|$
- Symmetry $S_2(\rho_A) = S_2\left(\rho_{X\setminus A}\right)$ is a manifestation of Schmidt decomposition of a pure state: ρ_A and $\rho_{X\setminus A}$ have identical sets of nonzero eigenvalues
- For $|A| < |X|/2$, ρ_A is close to the maximally mixed state: $S_2(\rho_A) \approx |A| \log d$