Solution to Factorable Multipoint Boundary Value Problems for fourth-Order Difference Equations

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Abstract. The paper deals with the establishment of solvability criteria and the construction of the unique solution in closed-form of factorable boundary value problems for fourth-order linear difference equations with constant coefficients subject to multipoint boundary conditions. The method proposed is a direct matrix procedure which is easily programmable.

Introduction

Many problems in sciences and applied mathematics end to the solution of higher-order difference equations [1]-[3]. The factorization method, in the cases where it can be applied, seems to be an efficient tool in solving nonlocal (multipoint) boundary value problems for difference equations [4]-[7]. Here we elaborate on the factorization and solution of a class of boundary value problems for fourth-order difference equations with multipoint boundary conditions.

Let S be the linear space of all real-valued functions (sequences) $y_k = y(k),\ k \in \mathbb{N}$. Consider the fourth-order difference equation

$$u_{k+4} + b_1 u_{k+3} + b_2 u_{k+2} + b_3 u_{k+1} + b_4 u_k = f_k, \tag{1}$$

where the coefficients $b_i \in \mathbb{R},\ i = 1, \ldots, 4$ with $b_4 \neq 0,\ f_k = f(k) \in S$ is the input function, and $u_k = u(k) \in S$ is the unknown function which satisfies (1) and the nonhomogeneous initial conditions

$$u_i = \beta_i, \quad i = 1, \ldots, 4, \tag{2}$$

where $\beta_i, \in \mathbb{R},\ i = 1, \ldots, 4$, or the nonlocal multipoint boundary conditions

$$\begin{aligned} \mu_{i1} u_1 + \mu_{i2} u_2 + \ldots + \mu_{il} u_l &= 0, \quad i = 1, 2, \quad l > 1, \\ \nu_{i1} u_1 + \nu_{i2} u_2 + ... + \nu_{is} u_s &= 0, \quad i = 1, 2, \quad 1 < l \leq s, \end{aligned} \tag{3}$$

where $\mu_{1j}, \mu_{2j} \in \mathbb{R}$, $j = 1, \ldots, l$ and $\nu_{1j}, \nu_{2j} \in \mathbb{R}$, $j = 1, \ldots, s$.

In Section 1, we examine the factorability of the initial value problem (1), (2) and derive an exact solution formula. In Section 2, we deal with the decompositionability and solution in closed-form of the boundary value problem (1), (3). Finally, we close by quoting some conclusions.

1. Initial value problem

Let the linear operators $B : S \to S$ be defined by

$$Bu_k = u_{k+4} + b_1 u_{k+3} + b_2 u_{k+2} + b_3 u_{k+1} + b_4 u_k, \tag{4}$$

with

$$b_3 = \frac{1}{2} b_1 \left(b_2 - \frac{1}{4} b_1^2 \right), \quad b_4 = \frac{1}{4} \left(b_2 - \frac{1}{4} b_1^2 \right)^2, \quad b_1^2 \neq 4b_2, \quad b_1, b_2 \in \mathbb{R}. \tag{5}$$

Let $\widehat{B} : S \to S$ be a restriction of B on

$$D(\widehat{B}) = \{ u_k \in S : u_i = \beta_i, \quad i = 1, ..., 4 \}, \tag{6}$$

where $\beta_i \in \mathbb{R},\ i = 1, \ldots, 4$.

Furthermore, let the linear operator $A : S \to S$ be defined by

$$Au_k = u_{k+2} + a_1 u_{k+1} + a_2 u_k, \tag{7}$$

where

$$a_1 = \frac{1}{2} b_1, \quad a_2 = \frac{1}{2} \left(b_2 - \frac{1}{4} b_1^2 \right). \tag{8}$$

Let $\widehat{A}_1, \widehat{A}_2 : S \to S$ be two restrictions of A defined, respectively, as

$$\begin{aligned} \widehat{A}_1 y_k &= y_{k+2} + a_1 y_{k+1} + a_2 y_k, \\ D(\widehat{A}_1) &= \{ y_k \in S : y_1 = \beta_3 + a_1 \beta_2 + a_2 \beta_1, \quad y_2 = \beta_4 + a_1 \beta_3 + a_2 \beta_2 \}, \end{aligned} \tag{9}$$

and

$$\begin{aligned} \widehat{A}_2 u_k &= u_{k+2} + a_1 u_{k+1} + a_2 u_k, \\ D(\widehat{A}_2) &= \{ u_k \in S : u_i = \beta_i, \quad i = 1, 2 \}. \end{aligned} \tag{10}$$

Lemma 1. *The operator $\widehat{B}$ is decomposed into*

$$\widehat{B} = \widehat{A}_1 \widehat{A}_2. \tag{11}$$

Let $u_k^{(1)}, u_k^{(2)}$ be a fundamental set of solutions of the homogeneous equation

$$Au_k = u_{k+2} + a_1 u_{k+1} + a_2 u_k = 0, \tag{12}$$

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$$C_0 = \begin{pmatrix} u_1^{(1)} & u_1^{(2)} \\ u_2^{(1)} & u_2^{(2)} \end{pmatrix}, \tag{13}$$

and $u_k^{(y_k)},\ \ y_k^{(f_k)}$ be partial solutions of the nonhomogeneous equations

$$Au_k = y_k, \quad Ay_k = f_k, \tag{14}$$

respectively, where $y_k, f_k \in S$.

Theorem 1. *The initial value problem*

$$\widehat{B}u_k = f_k, \tag{15}$$

for every $f_k \in S$, admits exactly one solution given by,

$$u_k = \widehat{A}_2^{-1} y_k = u_k^{(y_k)} - \left(u_k^{(1)},\ u_k^{(2)}\right) C_0^{-1} \begin{pmatrix} \beta_1 - u_1^{(y_k)} \\ \beta_2 - u_2^{(y_k)} \end{pmatrix}, \tag{16}$$

where

$$y_k = \widehat{A}_1^{-1} f_k = y_k^{(f_k)} - (u_k^{(1)},\ u_k^{(2)}) C_0^{-1} \begin{pmatrix} \beta_3 + a_1\beta_2 + a_2\beta_1 - y_1^{(f_k)} \\ \beta_4 + a_1\beta_3 + a_2\beta_2 - y_2^{(f_k)} \end{pmatrix}. \tag{17}$$

2. Multipoint boundary value problem

Let now $\widehat{B}_m$ be a restriction of the operator B defined on

$$D(\widehat{B}_m) \quad = \quad \{u_k \in S : M\mathbf{u}_l = 0, \quad N\mathbf{u}_s = 0\} \tag{18}$$

where

$$\begin{aligned} M &= \begin{pmatrix} \mu_{11} & \mu_{12} & \cdots & \mu_{1l} \\ \mu_{21} & \mu_{22} & \cdots & \mu_{2l} \end{pmatrix}, \\ N &= \begin{pmatrix} \nu_{11} & \nu_{12} & \cdots & \nu_{1s} \\ \nu_{21} & \nu_{22} & \cdots & \nu_{2s} \end{pmatrix} \\ &= \begin{pmatrix} 0 & \mu_{11}a_1 & \mu_{11}+\mu_{12}a_1 & \mu_{12}+\mu_{13}a_1 & \cdots & \mu_{1,l-1}+\mu_{1l}a_1 & \mu_{1l} \\ 0 & \mu_{21}a_1 & \mu_{21}+\mu_{22}a_1 & \mu_{22}+\mu_{23}a_1 & \cdots & \mu_{2,l-1}+\mu_{2l}a_1 & \mu_{2l} \end{pmatrix}, \end{aligned} \tag{19}$$

and

$$\mathbf{u}_l = \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_l \end{pmatrix}, \quad \mathbf{u}_s = \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_s \end{pmatrix}, \tag{20}$$

with $s = l + 2$.

Further, let $\widehat{A}_m : S \to S$ be a restriction of A defined as

$$\begin{aligned} \widehat{A}_m u_k &= u_{k+2} + a_1 u_{k+1} + a_2 u_k, \\ D(\widehat{A}_m) &= \{u_k \in S : M\mathbf{u}_l = 0\}. \end{aligned} \tag{21}$$

Lemma 2. *The operator $\widehat{B}_m$ is the self-composition*

$$\widehat{B}_m = \widehat{A}_m^2. \tag{22}$$

Let

$$\mathbf{y}^{(f_k)} = \begin{pmatrix} y_1^{(f_k)} \\ y_2^{(f_k)} \\ \vdots \\ y_l^{(f_k)} \end{pmatrix}, \quad \mathbf{u}^{(y_k)} = \begin{pmatrix} u_1^{(y_k)} \\ u_2^{(y_k)} \\ \vdots \\ u_l^{(y_k)} \end{pmatrix}, \quad U = \begin{pmatrix} y_1^{(1)} & y_1^{(2)} \\ y_2^{(1)} & y_2^{(2)} \\ \vdots & \vdots \\ y_l^{(1)} & y_l^{(2)} \end{pmatrix}. \tag{23}$$

Theorem 2. *If*

$$\det MU \neq 0, \tag{24}$$

then the operator $\widehat{B}_m$ is uniquely and everywhere solvable on S and the unique solution of the nonlocal problem

$$\widehat{B}_m u_k = f_k, \tag{25}$$

for every $f_k \in S$, is given by

$$u_k = \widehat{A}_m^{-1} y_k = u_k^{(y_k)} - \left(y_k^{(1)}\ y_k^{(2)}\right) (MU)^{-1} M \mathbf{u}^{(y_k)}, \tag{26}$$

where

$$y_k = \widehat{A}_m^{-1} f_k = y_k^{(f_k)} - \left(y_k^{(1)}\ y_k^{(2)}\right) (MU)^{-1} M\ \mathbf{y}^{(f_k)}. \tag{27}$$

Example 1. *Find the solution of the boundary value problem*

$$u_{k+4} - 6u_{k+3} + 13u_{k+2} - 12u_{k+1} + 4u_k = k, \tag{28}$$

$$\begin{aligned} u_1 - 3u_2 + u_3 &= 0, \\ u_1 - 5u_2 + 2u_3 &= 0, \\ -3u_2 + 10u_3 - 6u_4 + u_5 &= 0, \\ -3u_2 + 16u_3 - 11u_4 + 2u_5 &= 0. \end{aligned} \tag{29}$$

The coefficients b_i, $i = 1, \ldots, 4$, fulfill (5) and the boundary conditions satisfy (19). Therefore the corresponding operator $\widehat{B}_m : S \to S$ can be factored as in (22) where $a_1 = -3$ and $a_2 = 2$. Then from Theorem 2 follows that the unique solution of (28), (29) is

$$u_k = \widehat{B}_m^{-1} k = 2^{k-1}(k-7) + \frac{k^3}{6} + \frac{k^2}{2} + \frac{4k}{3} + 5. \tag{30}$$

Conclusion

A factorization method for obtaining the closed-form solution to a class of factorable multipoint boundary value problems for fourth-order difference equations has been presented. The technique is easy to implement to any Computer Algebra System (CAS) and is economic and efficient compared to other existing methods.

References

[1] M. Foupouagnigni, W. Koepf, and A. Ronveaux, *On fourth-order difference equations for orthogonal polynomials of a discrete variable: derivation, factorization and solutions*, J. Differ. Equations Appl., **9**, pp. 777-804, 2003. https://doi.org/10.1080/1023619031000097035

[2] S. Stević, *General solution to a higher-order linear difference equation and existence of bounded solutions* Adv. Differ. Equ. **2017**, 377 2017. https://doi.org/10.1186/s13662-017-1432-7

[3] G. Papaschinopoulos, C.J. Schinas, and G. Ellina, *On the dynamics of the solutions of a biological model*, J. Differ. Equations Appl., **20**, pp. 694–705, 2014. https://doi.org/10.1080/10236198.2013.806493

[4] I.N. Parasidis, and E. Providas, *Closed-Form solutions for some classes of loaded difference equations with initial and nonlocal multipoint conditions*, In: Daras N., Rassias T. (eds) Modern Discrete Mathematics and Analysis, Springer Optimization and Its Applications, vol 131, pp. 363–387, Springer, Cham, 2018. https://doi.org/10.1007/978-3-319-74325-7_19

[5] I.N. Parasidis, and E. Providas *An exact solution method for a class of nonlinear loaded difference equations with multipoint boundary conditions*, J. Differ. Equations Appl., **24**, pp. 1649–1663, 2018. https://doi.org/10.1080/10236198.2018.1515928

[6] I.N. Parasidis, and P. Hahamis, *Factorization method for solving multipoint problems for second order difference equations with polynomial coefficients*, In: Raigorodskii A.M., Rassias M.T. (eds) Discrete Mathematics and Applications, Springer Optimization and Its Applications, vol 165, Springer, Cham, 2020. https://doi.org/10.1007/978-3-030-55857-4_17

[7] E. Providas, *Operator factorization and solution of second-order nonlinear difference equations with variable coefficients and multipoint constraints*, In: Rassias T.M., Pardalos P.M. (eds) Nonlinear Analysis and Global Optimization, Springer Optimization and Its Applications, vol 167, Springer, Cham, 2021. https://doi.org/10.1007/978-3-030-61732-5_20

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