On the number of knots with a given arc index

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Abstract. We show that the number of knots with arc index at most N is at least x^x for $x = N/24$.

We study the structure and statistical characteristics of the set of classical knots (see [M15, M18, M18b, M19, BM19, M20] and references therein). A particular point of our study is the growth rate of the number of knots with respect to various complexity measures on the set of knots.

Historically, the crossing number is considered as the most natural knot complexity measure, and to date, there is an extensive series of results on the growth rate of the number of knots with respect to the crossing number. This topic is studied in particular in [ES87, W92, STh98, **?**, St04, **?**, M20].

In a new line of research, we consider the growth rate of the number of knots with respect to the arc index (definitions and properties of the arc index are given in [Cr95, Cr95b, CN96, BP00, Dy06, JL12]).

As it turns out, the methods developed while studying the growth rate with respect to the crossing number (these methods are based on properties of alternating links, the Tait conjectures, properties of the Jones polynomial, and Tutte's results on the number of planar maps) give only very weak estimates for the growth rate with respect to the arc index.

To obtain better estimates with respect to the arc index, we develop two new approaches. Our first approach is based on the analysis of the structure of conjugacy classes in braid groups, the second one is based on using JSJ decomposition properties to construct embeddings of pure braid groups into the set of prime knots. With the second approach, we have obtained the following result.

Theorem 1. *The number of prime knots with arc index at most N is at least x^x for $x = N/24$.*

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