Linearizability property of Lie symmetry algebra

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In honor of Vladimir Gerdt

Rectification theorem

System of first-order differential equations

$$\frac{dx}{dt} = f(x), x = (x_1, x_2, ..., x_n)$$

could be mapped in some neighborhood U of point $x = u$ by diffeomorphism into

$$\frac{dz}{dt} = const,$$

if $f(u) \neq 0$ (generic case). It is well-known rectification theorem of vector fields.

Poincare theorem

The situation becomes more complicated if $f(u) = 0$ (degenerate case). Poincare showed that in this case it is important to analyse Jacobian of f defined as

$$J = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \dots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \dots & \frac{\partial f_2}{\partial x_n} \\ \dots & \dots & \dots & \dots \\ \frac{\partial f_n}{\partial x_1} & \frac{\partial f_n}{\partial x_2} & \dots & \frac{\partial f_n}{\partial x_n} \end{bmatrix}$$

Hence, if matrix $J(u)$ is diagonalizable, then non-resonant condition for eigenvalue implies linearization to the form

$$\frac{dz}{dt} = Az, A = const.$$

Exact linearization

Typically in applied mathematics the phrase linearization merely means

"approximating of f by its linear part in Taylor series".

Here in contrast we mean completely different.

We consider ODEs of the form:

$$y^{(n)} = f(x, y, y', \ldots, y^{(n-1)}), \quad n \geq 2 \tag{1}$$

and want to know if Eq. (1) is linearizable through some transformation.

$$y^{(n)} = f(x, y, y', \ldots, y^{(n-1)})$$

$$\downarrow$$

$$u^{(n)}(t) + \sum_{k=0}^{n-1} a_k(t)\, u^{(k)}(t) = 0$$

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Point transformation

Point transformation (PT) is an analytical diffeomorphism:

$$t = t(x, y)\,, u = u(x, y)\,, t_x u_y - t_y u_x \neq 0.$$

$t = \sqrt{x}\,, u = y\,, x \neq 0.$

$$y''' + 3y''/(2x) = 0$$

$$\downarrow$$

$$u''' = 0$$

Algebra of symmetry generators

Assume that $(t, u) = T_\alpha(x, y) = (f(x, y, \alpha), g(x, y, \alpha))$ is one-parameter symmetry group with a parameter α such that it transform vector field of (1) into itself.

It is defined uniquely by its infinitesimal generator

$$(\xi(x, y), \eta(x, y)) = (f_\alpha(x, y, \alpha), g_\alpha(x, y, \alpha))|_{\alpha=0}.$$

Set of $\mathcal{X} = (\xi, \eta)$ forms Lie algebra

$$[\mathcal{X}_i, \mathcal{X}_j] = \sum_{k=1}^{m} c_{ij}^k \mathcal{X}_k\,, \quad 1 \le i < j \le m\,, \tag{2}$$

where c_{ij}^k is called structure constants and m is dimension of Lie algebra.

Main theorem

Theorem 1 [Lyakhov et al, ISSAC'2017].

Eq. (1) with $n \geq 2$ is linearizable by a point transformation if and only if one of the following conditions is fulfilled:

1. $n = 2$, $m = 8$
2. $n \geq 3$, $m = n + 4$;
3. $n \geq 3$, $m \in \{n + 1, n + 2\}$ and the derived algebra of L is abelian of dimension n.

Lie Algebras of Vector Fields [Lisle et al, JSC'2017]

1. Toolkit of algorithms for determining systems, with the goal of extracting algebraic information (isomorphism invariants) about the Lie algebra, and geometric information $\rightarrow$ diffeomorphism invariants (orbit dimension, transitivity, and structure of isotropy algebra) about the vector fields that constitute it
2. As well as decision procedures (for example, testing whether such a system really defines a Lie algebra), it allows to extract determining systems for the vector fields that make up various structural parts such as the derived algebra, solvable radical, centre, etc.
3. None of these methods require solving the system.

Constant coefficient case

$$u^{(n)}(t) + \sum_{k=0}^{n-1} a_k\, u^{(k)}(t) = 0$$

Let suppose all characteristic numbers λ are different. Then, derived algebra of fundamental solution consists of operators

$$\mathcal{X}_i = exp(\lambda_i t)\frac{\partial}{\partial u},\ i = 1...n$$

Shift operator

$$\mathcal{X}_{n+1} = \frac{\partial}{\partial t} \Rightarrow [\mathcal{X}_{n+1}, \mathcal{X}_i] = \lambda_i \mathcal{X}_i$$

Constant coefficient case

For arbitrary non-degenerate matrix T_{ik}

$$\bar{\mathcal{X}}_i = \sum_{k=1}^{n} T_{ik}\mathcal{X}_k,$$

Then

$$[\mathcal{X}_{n+1}, \bar{\mathcal{X}}_i] = \sum_{k=1}^{n} T_{ik}\lambda_k\mathcal{X}_k \Rightarrow \sum_{k} T_{ik}\lambda_k T_{kj}^{-1} = A_{ij}$$

matrix A admits the same characteristic polynomial

$$(\lambda - \lambda_1)\cdot(\lambda - \lambda_2)\cdot \ldots \cdot(\lambda - \lambda_n)$$

Bluman-Kumei equations

Assume that a point transformation $(X, Y) = T(x, y)$ maps Eq. (1) to:

$$Y^{(n)} = F(X, Y, Y', \ldots, Y^{(n-1)})$$

Then T induces an isomorphism between their point symmetry algebras,

$$T : a\,\partial_x + b\,\partial_y \mapsto A\,\partial_X + B\,\partial_Y,$$

which is expressed as Bluman-Kumei equations:

$$a(x, y)X_x + b(x, y)X_y = A(x, y)\,,$$
$$a(x, y)Y_x + b(x, y)Y_y = B(x, y)\,,$$

Remark: In B-K equations, new symmetry (A, B) is expressed in old variables.

Derived algebra

Except trivializable case, derived algebra is abelian subalgebra. When equation is linear, infinitesimal generators satisfy

$$\xi = 0, \ \frac{\partial}{\partial u}\eta(t,u) = 0$$

Substitution of $(t,u) \mapsto (x,y)$ in terms of B-K gives explicit expression of nonlinear PDE system for linearizing mapping.

Theorem. Linearizing differential system admits finite-dimensional construction.

Levi algebra

In the case $n \geq 3$, $m = n + 4$, we have the radical[1]

$$\mathcal{R} = [\frac{\partial}{\partial u}, t\frac{\partial}{\partial u}, ..., t^{n-1}\frac{\partial}{\partial u}, u\frac{\partial}{\partial u}].$$

One can check the above result by using the maximal property of the radical. Further, the derived algebra of $\mathcal{R}$ is an $n-$dimensional abelian algebra

$$[\mathcal{R}, \mathcal{R}] = [\frac{\partial}{\partial u}, t\frac{\partial}{\partial u}, ..., t^{n-1}\frac{\partial}{\partial u}]$$

which is generated by the fundamental solution set.
It immediately gives us the algorithm for reconstruction of linearizing mapping based on Levi decomposition.

[1] A radical in $\mathcal{L}$ is the largest solvable ideal.

Contact transformation

Add one more variable $p = y'(x)$ in point transformation:

$$X = X(x, y, p)\,, Y = Y(x, y, p)\,, P = Y'(X) = Y_p/X_p.$$

The last expression is coming from $Y_p = (Y(X))_p = X_p Y'(X)$.
Another expression for P is from total differentiation

$$P = Y'(X) = D_x Y/D_x X = (Y_x + pY_y + p'Y_p)/(X_x + pX_y + p'X_p).$$

Thus $X_p(Y_x + pY_y) = Y_p(X_x + pX_y)$ is also required to equate above two expressions.
And the nonsingularity of Jacobian can be simplified as
$(PX_y - Y_y)((P_x + pP_y)X_p - (X_x + pX_y)P_p) \neq 0$.

$X = p\,, Y = xp - y\,, P = x.$

$$y''' - 3y''^2/(2y') = 0$$

$$\downarrow$$

$$Y''' + 3Y''/(2X) = 0$$

Generalizations, contact symmetry

Above theory admits generalization on basis of two following theorems:

Theorem 2 [Yumaguzhin,1996].

The dimension of the contact symmetry algebra of any 3rd order linear ODE is equal to one of the following numbers: 4, 5, and 10.

Theorem 3 [Svirshchevskii,1995].

A linear ODE of kth order with $k \geq 4$ does not possess nontrivial (non-point) contact symmetries.

Generalizations, system of 2 ODE

$$x''(t) = f(t, x, y, x', y')$$
$$y''(t) = g(t, x, y, x', y')$$

We denote the of symmetry algebra of system by $\mathcal{L}$, the abelian algebra by $\mathcal{A} = [X_1, X_2, X_3, X_4]$ where X_i is associated with a fundamental solution $(f_i(t), g_i(t))$, also let $\mathcal{D} = [\mathcal{L}, \mathcal{L}]$, $\mathcal{D}' = [\mathcal{D}, \mathcal{D}]$, $n = \dim(\mathcal{L})$ and $m = \dim(\mathcal{D})$.

1. If $n \in \{5, 6\}$ then $\mathcal{A} = \mathcal{D}$;
2. If $n = 7$ and $\dim(\mathcal{D}) = 4$ then $\mathcal{A} = \mathcal{D}$;
3. If $n = 7$ and $\dim(\mathcal{D}) > 4$ then $\mathcal{A} = [x\partial_x + y\partial_y, \mathcal{D}]$;
4. If $n = 8$, then $\mathcal{A} = \{X \in \mathcal{L} : [X, Y] = 0, \forall Y \in \mathcal{D}'\}$.
5. If $n = 15$

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