On the Bures metric and quantum Fisher information for rank-deficiency qudit states

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Recent developments in the emerging field of quantum computation and quantum information theory have received a great deal of attention to studies of $N-$level systems. According to the statistical interpretation of quantum theory, it is a new kind of a probabilistic model and quantum $N-$level system is an analogue of the classical probability model with a finite probabilistic space [1]. Despite being the simplest quantum system, it turns out that this quantum probabilistic model drastically differs from its classical counterpart. The quantum analogue of classical probability distribution is the state of system, described by a density matrix $\varrho \in \mathfrak{P}_N$. The state space $\mathfrak{P}_N$ comprises all $N \times N$ Hermitean, normalized semi-positive matrices,

$$\mathfrak{P}_N = \{\, X \in M_N(\mathbb{C}) \,|\, X = X^\dagger\,,\ X \geq 0\,,\ \mathrm{Tr} X = 1 \,\}\,. \qquad (1)$$

The analogue of a classical random variable is a quantum observable, represented by an $N \times N$ Hermitian matrix A. If $A = \sum_i a_i |e_i\rangle\langle e_i|$ is the spectral decomposition of A with respect to the orthonormal basis of eigenvectors $|e_i\rangle$ corresponding to eigenvalues, $\mathrm{spec}(A) = \{a_1, a_2, \dots, a_N\}$, then the *probability distribution of the observable A in state ϱ over spectrum* is defined as:

$$p^A_\varrho(a_i) = \mathrm{Tr}\Big(\varrho\, |e_i\rangle\langle e_i|\Big)\,, \qquad i = 1, 2, \dots, N\,. \qquad (2)$$

Following this analogy, many methods from information geometry, the geometry of manifolds of probability distributions in classical statistics [2], have been adopted for various studies of quantum systems [3]. Particularly, an issue of establishing of Riemannian structures on the quantum counterparts of space of probability measures, became a subject of recent investigations. Nowadays, due to practical goals in the area of modern quantum technologies, a special attention has been drawn to the question of the determination of quantum analogues of a well-known, natural Riemannian metric, the so-called Fisher metric, as well as a family of affine connections.

Our report is devoted to the discussion of this topic within quantum information geometry. We will analyse the metric on $\mathfrak{P}_N$, originated from the distance function $d(\varrho_1, \varrho_2)$ between density matrices $\varrho_1, \varrho_2 \in \mathfrak{P}_N$, which is known under different names, the Fisher (in statistics), Bures (classical and quantum information theory), Wasserstein metric (optimal transport):

$$d(\varrho_1, \varrho_2) := \sqrt{\mathrm{tr}\varrho_1 + \mathrm{tr}\varrho_2 - 2\mathrm{tr}\left(\varrho_1^{1/2}\,\varrho_2\,\varrho_1^{1/2}\right)^{1/2}}\,. \tag{3}$$

The distance function (3) corresponds to the Bures metric which belongs to the special class of the so-called monotone Riemannian metrics. It is minimal among all monotone metrics and its extension to pure states is exactly the Fubini–Study metric [4].

Explicit formulae for the Bures metric are known for special cases. Particularly, Dittmann has derived several explicit formulae (that do not require any diagonalization procedure) for the Bures metric on the manifold of finite-dimensional nonsingular density matrices [5, 6, 7, 8].

However, owing to the nontrivial differential geometry of the state space $\mathfrak{P}_N$, studies of its Riemannian structures require a refined analysis for the non maximal rank density matrices. Indeed, let $\mathfrak{P}_{N,k} \subset \mathrm{M}_N(\mathbb{C})$ be a manifold comprising of the normalized $N \times N$ density matrices of rank k. According to [8], every $\mathfrak{P}_{N,k}$ admits the Bures metric g_B and hence one can consider subspace of fixed rank as the Riemannian manifold $(\mathfrak{P}_{N,k}, \mathrm{g}_B)$. The union $\mathfrak{D}_N := \bigcup_{k=1}^{N} \mathfrak{P}_{N,k}$ is not a manifold, but a convex subset of affine space of all normalized Hermitian matrices. Furthermore, $\mathfrak{D}_N$ cannot be isometrically embedded to any manifold. Due to these reasons the interplay between the Bures metric and the quantum Fisher information deserves a special analysis (see e.g. the discussion in [9]).

In the present report, following the purification method and the fundamental principle of parallel transport along the state (see e.g. [5]) we will derive the metric on each strata $\mathfrak{P}_{N,k}$ from the the bundle projection $M_N(\mathbb{C}) \to \mathfrak{P}_N$. Our exposition is exemplified in details by considering the Bures metric for qubit ($N = 2$) and qutrit ($N = 3$) in different strata.

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