Quaternionic description of the Euler trajectories of a rigid body in various configuration spaces.

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Abstract. The paper considers some properties of the Euler rotations of a rigid body in various configuration spaces of rotations. It is shown that the trajectories of these rotations are specified using spherical linear interpolation on to $\mathbb{R}^4$. Their images are found in the configuration manifold of rotations in the form of a ball of radius in $\mathbb{R}^3$. It is proved that, depending on the initial and final position, the trajectories of the Euler turns in $\mathbb{R}^3$ are either plane curves, or straight line segments, or circular arcs.

Introduction

One of the possible options for describing the configuration space of rotations is a group $SO(3)$ - a group of special orthogonal matrices [1]. The elements of this group are matrices of direction cosines that determine the position of the axes connected with a rigid body relative to some fixed coordinate system. Along with Euler angles and airplane angles, parameterization of turns using direction cosines is a fairly common tool for describing rotations of a rigid body, however, in many cases, significant advantages in describing rotations of a rigid body can be achieved by using unit quaternions $\mathrm{q} = [q_0, q_1, q_2, q_3]$, $q_0^2 + q_1^2 + q_2^2 + q_3^2 = 1$. Moreover, the group of unit quaternions $Sp(1)$ covers the group $SO(3)$ in a two-fold manner [2]. Those, each element of the group corresponds to two elements of the group $Sp(1)$. Quaternions $[q_0, q_1, q_2, q_3]$ and $[-q_0, -q_1, -q_2, -q_3]$ define the same rotation of a rigid body. In this case, the configuration space of turns is the sphere $S^3: \ q_0^2 + q_1^2 + q_2^2 + q_3^2 = 1$ - a hypersphere of a unit radius in four-dimensional space $\mathbb{R}^4$.

Establishing connections between various admissible parameterizations of a group of rotations is one of the most important problems in the kinematics of an absolutely rigid body. This is due to the fact that, depending on the specific problem being solved in this practically important area of mechanics, one or another parameterization of the rotation group turns out to be convenient [3]. One of the ways to parametrize a group of rotations is based on the fact that a sphere S^3 can be mapped into a ball with a radius π of three-dimensional Euclidean space $\mathbb{R}^3$ [4].

The resulting mapping of a group $SO(3)$ into a ball of radius π is not bijective, since opposite points of the ball diameters correspond to the same direction cosine matrix. However, the resulting configuration manifold can be used to obtain visual trajectories of rotations of a rigid body, in particular, its Euler rotations.

According to the Euler-D'Alembert theorem, a rigid body in its spherical motion can be transferred from one position to another by one rotation around a fixed axis. Such turns are called end, flat, or Eulerian. In various works, for example [3, 5], it is noted that plane turns are associated with geodesics on the hypersphere S^3, which endows them with some extremal properties used in solving applied problems.

The goal of this work is to study some properties of the trajectories of Euler turns in various configuration spaces, namely, on a hypersphere S^3 and in a three-dimensional ball with radius π.

1. Construction of trajectories of Euler turns on the sphere S^3

Let choose two arbitrary points $\mathrm{q}^{(1)} \in S^3$, $\mathrm{q}^{(2)} \in S^3$ and connect them by a curve, using for this the concept of linear spherical interpolation of quaternions [6]:

$$\mathrm{q}^{(12)}(u) = \frac{\sin(\arccos(\mathrm{q}^{(1)} \cdot \mathrm{q}^{(2)})(1-u))}{\sin(\arccos(\mathrm{q}^{(1)} \cdot \mathrm{q}^{(2)}))}\mathrm{q}^{(1)} + \frac{\sin(\arccos(\mathrm{q}^{(1)} \cdot \mathrm{q}^{(2)})u)}{\sin(\arccos(\mathrm{q}^{(1)} \cdot \mathrm{q}^{(2)}))}\mathrm{q}^{(2)}, \quad (1)$$

where $u \in [0, 1]$.

The authors have proved the following theorems.

Theorem 1.1

The curve defined by spherical linear interpolation (1) is a geodesic line on the surface of the hypersphere S^3.

Theorem 1.2

Movement along a curve defined by spherical interpolation (1) corresponds to an Euler (planar) rotation of a rigid body.

Theorem 1.3

When moving along a curve defined by spherical interpolation (1), the unit direction vector of the rotation axis is determined by the formula

$$\vec{e} = \frac{1}{\sqrt{1-(\mathrm{q}^{(1)} \cdot \mathrm{q}^{(2)})^2}}(q_0^{(1)}\vec{q}^{\,(2)} - q_0^{(2)}\vec{q}^{\,(1)} + \vec{q}^{\,(1)} \times \vec{q}^{\,(2)})$$

2. Construction of trajectories of Euler turns in $\mathbb{R}^3$

To solve applied problems of motion control, it is convenient to go from completely deprived of visualization of specifying a trajectory on a hypersphere in four-dimensional space to specifying it in a ball with a radius in a three-dimensional Euclidean space, since a set of unit quaternions can be associated with points of a ball of radius. To each unit quaternion

$$\mathrm{q} = \cos\frac{\chi}{2} + \sin\frac{\chi}{2}\vec{e},$$

corresponds to a point of a three-dimensional sphere with a radius vector

$$\vec{r} = \chi\vec{e}.$$

The connection between the components of the quaternion and the coordinates of points in the ball can be written by the equalities [7]:

$$\begin{cases} q_0 = \cos\dfrac{\sqrt{x_1^2+x_2^2+x_3^2}}{2}, \\ q_k = \dfrac{x_k}{\sqrt{x_1^2+x_2^2+x_3^2}}\sin\dfrac{\sqrt{x_1^2+x_2^2+x_3^2}}{2},\ k=1,\ 2,\ 3. \end{cases}$$

whence,

$$x_k = \frac{2q_k\arccos q_0}{\sqrt{1-q_0^2}},\ k=1,\ 2,\ 3. \tag{2}$$

It is worth noting separately the case $q_0 = 1$, for this case the vector $\vec{e}$ of the unit quaternion is determined ambiguously, therefore the functions x_k obtained above have a discontinuity, however, it can be eliminated by setting

$$x_k|_{q_0=1,q_1=0,q_2=0,q_3=0} = \lim_{q_0\to 1,q_1\to 0,q_2\to 0,q_3\to 0} x_k = 0.$$

We apply mapping (2) to the trajectory of the Euler rotation (1), the resulting image will be denoted by

$$\vec{r}^{\,(12)}(u) = [x_1^{(12)}(u), x_2^{(12)}(u), x_3^{(12)}(u)]. \tag{3}$$

For trajectories of Euler turns (3), the authors proved the following theorems.

Theorem 2.1

The trajectory of the Euler rotation (3) is a plane curve, and the plane of this curve passes through the origin.

Theorem 2.2

If the quaternions $\mathrm{q}^{(1)}$, $\mathrm{q}^{(2)}$ have zero real parts, then the trajectory of the Euler rotation (3) is a circular arc.

Theorem 2.3

If the quaternions $\mathrm{q}^{(1)}$, $\mathrm{q}^{(2)}$ have collinear vector parts, then the trajectory of the Euler rotation (3) is a straight line segment passing through the origin.

Conclusion

In this work, some properties of trajectories of Euler turns on the hypersphere S^3 were obtained. Also, a mapping was applied to the trajectories under consideration, which allows you to get their visual representation in the form of images in $\mathbb{R}^3$. It is proved that the images of the trajectories of the Euler turns in $\mathbb{R}^3$, are a plane curve, the plane of which passes through the origin. It is shown that for specially selected orientations $\mathrm{q}^{(1)}$, $\mathrm{q}^{(2)}$ of a rigid body, this curve is an arc of a circle or a straight line segment.

This result can be useful in solving problems of optimal control of the rotation of a rigid body.

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