Tropical Jacobian Conjecture

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Classical Jacobian Conjecture

Polynomial map $F(x) := (f_1(x), \ldots, f_n(x)) : K^n \to K^n$ where polynomials $f_1, \ldots, f_n \in K[X_1, \ldots, X_n]$ and K is a field of characteristic 0. Let F be invertible, so for some polynomials $g_1, \ldots, g_n \in K[X_1, \ldots, X_n]$ the map $G := (g_1, \ldots, g_n) : K^n \to K^n$ is inverse to F, i. e. the composition $(F \circ G)(x) = x$. Then Jacobian $J(F) := \det Jac(F) \in K$.

Jacobian conjecture, Keller, 1939: if $J(F) \in K$ then F is an isomorphism and its inverse is also a polynomial map.

Jacobian conjecture: a local isomorphism (due to the Implicit Function Theorem) for polynomial maps implies a global isomorphism.

Theorem

*For an algebraically closed field K if F is injective then F is bijective. (**Ax, 1968; Grothendieck, 1966**)*

Model-theoretic proof: reduction to finite fields using Nullstellensatz.

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Example

(**Pinchuk, 1994**). When $K = \mathbb{R}$ the conclusion of f being an isomorphism is wrong under the assumption $J(f) > 0$.

Tropical semi-ring

Tropical semi-ring T is endowed with operations $\oplus$, $\otimes$.

If T is an ordered semi-group then T is a tropical semi-ring with inherited operations $\oplus := \min$, $\otimes := +$.

If T is an ordered (resp. abelian) group then T is a *tropical semi-skew-field* (resp. *tropical semi-field*) w.r.t. $\oslash := -$.

Examples • $\mathbb{Z}^+ := \{0 \le a \in \mathbb{Z}\}$, $\mathbb{Z}^+_\infty := \mathbb{Z}^+ \cup \{\infty\}$ are commutative tropical semi-rings. ∞ plays a role of 0, in its turn 0 plays a role of 1;

• $\mathbb{Z}$, $\mathbb{Z}_\infty$ are semi-fields;

• $n \times n$ matrices over $\mathbb{Z}_\infty$ form a non-commutative tropical semi-ring: $(a_{ij}) \otimes (b_{kl}) := (\oplus_{1 \le j \le n} a_{ij} \otimes b_{jl})$.

Tropical polynomials

Tropical monomial $x^{\otimes i} := x \otimes \cdots \otimes x$, $Q = a \otimes x_1^{\otimes i_1} \otimes \cdots \otimes x_n^{\otimes i_n}$, its *tropical degree* $\mathrm{trdeg} = i_1 + \cdots + i_n$. Then $Q = a + i_1 \cdot x_1 + \cdots + i_n \cdot x_n$.

Tropical polynomial $f = \bigoplus_j (a_j \otimes x_1^{i_{j1}} \otimes \cdots \otimes x_n^{i_{jn}}) = \min_j\{Q_j\}$;

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Tropical Algebraic Rational Functions

$\min\{P_1, \dots, P_k\} - \min\{Q_1, \dots, Q_l\}$ is a tropical algebraic rational function where $P_1, \dots, P_k, Q_1, \dots, Q_l$ are linear functions with rational coefficients. It is a piece-wise linear function, so one can partition $\mathbb{R}^n$ into a finite number of n-dimensional polyhedra on each of which this function is linear. Conversely, any piece-wise linear function can be represented in this form (up to rationality of the coefficients). More generally, one can assume the coefficients to be real.

How to replace the Jacobian for non-smooth tropical algebraic rational maps (=tropical maps) $f = (f_1, \dots, f_n) : \mathbb{R}^n \to \mathbb{R}^n$ where $f_1, \dots, f_n$ are tropical algebraic rational functions? If f is an isomorphism then its inverse f^{-1} is also a tropical map.

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How to replace the Jacobian for non-smooth tropical algebraic rational maps (=tropical maps) $f = (f_1, \ldots, f_n) : \mathbb{R}^n \to \mathbb{R}^n$ where $f_1, \ldots, f_n$ are tropical algebraic rational functions? If f is an isomorphism then its inverse f^{-1} is also a tropical map.

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Weak version of a Tropical Jacobian Conjecture

For a tropical map $f = (f_1, \dots, f_n) : \mathbb{R}^n \to \mathbb{R}^n$ and a point $p \in \mathbb{R}^n$ consider all n-dimensional polyhedra containing p on which f is linear, the $n \times n$ matrices (=Jacobian matrices) of these linear maps denote by $A_1, \dots, A_k$, then $J_i = \det(A_i)$, $1 \le i \le k$ are their Jacobians. The convex hull of $A_1, \dots, A_k$ denote by $\partial_p(f)$.

Proposition

If $\partial_p(f)$ does not contain a singular matrix for any $p \in \mathbb{R}^n$ then f is an isomorphism.

The proof relies on Clarke's theorem (1974) that f (being Lipschitz) is a local homeomorphism. Then being proper (= the preimage of every compact is again compact) f is a (global) homeomorphism.

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Non-necessity of the Weak Conjecture

A tropical polynomial isomorphism $f : \mathbb{R}^2 \to \mathbb{R}^2$ is a composition of a lower-triangular and an upper-triangular isomorphisms

$$(x, y) \mapsto (x, y + \min\{\alpha x, \beta x\}), \ \alpha < \beta,$$

$$(x, y) \mapsto (x + \min\{ay, by\}, y), \ a < b.$$

Then $f(x, y)$ is linear on 4 pieces:

$$f = (x + a(y + \alpha x), y + \alpha x) \quad \text{if} \quad x > 0, y + \alpha x > 0;$$

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Strong version of the tropical Jacobian conjecture

If a tropical map $f : \mathbb{R}^n \to \mathbb{R}^n$ is an isomorphism then all the Jacobians J_i have the same sign, say $J_i > 0$ for all i.

Theorem

If $f = (f_1, f_2) : \mathbb{R}^2 \to \mathbb{R}^2$ is a tropical polynomial map and all $J_i > 0$ then f is an isomorphism.

Example

A tropical rational map $g : (x, y) \to (|x| - |y|, |x + y| - |x - y|)$ has all the positive Jacobians $J_i > 0$, but $g(x, y) = g(-x, -y)$ is not an isomorphism. Modifying g one can construct a tropical polynomial map $\mathbb{R}^3 \to \mathbb{R}^3$ with all positive $J_i > 0$ being not an isomorphism.

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An algorithm to verify whether a tropical map is an isomorphism

A point $p \in \mathbb{R}^n$ is *regular* for a map $f : \mathbb{R}^n \to \mathbb{R}^n$ if for any $x \in f^{-1}(p)$ its Jacobian $J_f(x) \neq 0$. By Sard's lemma the set of regular values is dense.

Theorem

A necessary and sufficient condition for a tropical map $f : \mathbb{R}^n \to \mathbb{R}^n$ to be an isomorphism is that all the Jacobians J_i have the same sign and $|f^{-1}(p)| = 1$ for at least one regular value $p \in \mathbb{R}^n$.

An algorithm yields a partition of $\mathbb{R}^n = \cup_i P_i$ into polyhedra P_i such that f is linear on each P_i. Then any point $p \in \mathbb{R}^n \setminus \cup_i f(\partial P_i)$ is regular. The algorithm tests whether $|f^{-1}(p)| = 1$. All this can be performed invoking linear programming.

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A point $p \in \mathbb{R}^n$ is *regular* for a map $f : \mathbb{R}^n \to \mathbb{R}^n$ if for any $x \in f^{-1}(p)$ its Jacobian $J_f(x) \neq 0$. By Sard's lemma the set of regular values is dense.

Theorem

A necessary and sufficient condition for a tropical map $f : \mathbb{R}^n \to \mathbb{R}^n$ to be an isomorphism is that all the Jacobians J_i have the same sign and $|f^{-1}(p)| = 1$ for at least one regular value $p \in \mathbb{R}^n$.

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Tameness of tropical rational plane automorphisms

A *triangle* tropical rational plane automorphism has a form $(x, y) \to (x, y + \min\{ax, bx\})$, $a, b \in \mathbb{Z}$. A linear tropical rational automorphism has a form $(x, y) \to (ax + by, cx + dy)$, $a, b, c, d \in \mathbb{Z}$, $ad - bc = \pm 1$.

Proposition

The group of tropical rational homogeneous automorphisms is generated by triangular and linear automorphisms.

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