NEWTON POLYTOPES AND DIFFENTIAL ALGEBRA

Askold Khovanskii*

April 22, 2021

Abstract. Our main theorem provides a new result on nonintegrability of linear differential equations in finite terms. Its statement uses geometry of convex polytopes. It is much stronger than all known results of similar type. In particular it implies the classical criterium for solvability of such equations by quadratures. Its proof is based on classical ideas due to Liouville, Ritt and Rozenlicht and on arguments from Newton polyhedra theory and toric geometry.

*The work was partially supported by the Canadian Grant No. 156833-17.

I. LIOUVILLE'S–RITT'S AND ROSENLICHT'S THEOREMS ON SOLVABILITY OF DIFFERENTIAL EQUATIONS BY GENERALIZED QUADRATURES

Let K be some field of meromorphic functions on a connected domain on complex line which is closed under differentiation or and abstract differential field.

Definition 1. *An element y in a differential field F containing K is an integral of an element $a \in K$ if $y' = a$.*

An element y is an exponential of integral of $a \in K$ if $y' = ay$.

An extension $Q \supset K$ is representable by generalized quadratures if there is a chain of extensions

$$K = K_0 \subset K_1 \subset \cdots \subset K_m \supset G$$

such that for any $i = 1, \ldots, m$*:*

1) or $K_i \supset K_{i-1}$ *is the algebraic closure of* K_{i-1}*;*

2) or $K_i \supset K_{i-1}\langle y_i \rangle$ *is the algebraic closure of* $K_{i-1}\langle y_i \rangle$ *obtained from* K_{i-1} *by adding an integral of* $a \in K_{i-1}$*, i.e.* $y_i' = a$*;*

3) or $K_i \supset K_{i-1}\langle y_i \rangle$ *is the algebraic closure of* $K_{i-1}\langle y_i \rangle$ *obtained from* K_{i-1} *by adding an exponential of integral of* $a \in K_{i-1}$*, i.e.* $y_i' = a y_i$*.*

Consider a homogeneous linear differential equation

$$y^{(n)} + a_1 y^{(n-1)} + \cdots + a_n y = 0 \tag{1}$$

whose coefficients a_i belong to a differential field K.

Theorem 1. *If the equation (1) has a non zero solution representable by generalized quadratures over* K *then it necessarily has a solution of the form* $y_1 = \exp z$ *where* z' *is algebraic over* K.

The following lemma is well known and obvious.

Lemma 2. *Assume that the equation (1) has a non zero solution* y_1 *representable by generalized quadratures over* K.

Then the equation (1) can be solved by generalized quadratures over K *if and only if the linear differential equation of order* $(n-1)$ *over the differential field* $K\langle y_1'/y_1\rangle$ *obtained from (1) by the reduction of order using the solution* y_1 *is solvable by generalized quadratures over* $K\langle y_1'/y_1\rangle$.

Thus Theorem 1 provides the following criterium for solvability of the equation (1) by generalized quadratures.

Theorem 3. *The equation (1) is solvable by generalized quadratures over K if and only if the following conditions hold:*

1) the equation (1) has a solution y_1 of the form $y_1 = \exp z$ where $z' = f$ is algebraic over K;

2) the linear differential equation of order $(n-1)$ over $K\langle y_1'/y_1\rangle$ obtained from (1) by the reduction of order using the solution y_1 is solvable by generalized quadratures over $K\langle y_1'/y_1\rangle$.

The standard proof (E. Picard and E. Vessiot, 1910) of Theorem 1 uses the differential Galois theory and is rather involved (see [5]).

In the case when the equation (1) is a Fuchsian differential equation and K is the field of rational function of one complex variable Theorem 3 has a topological explanation (see [6]) which allows to prove much stronger version of this result. But in general case Theorem 3 does not have a similar visual explanation.

Maxwell Rosenlicht in 1973 proved [2] the following theorem.

Theorem 4. *Let n be a positive integer, and let Q be a polynomial in several variables with coefficients in a differential field K and of total degree less than n. Then if the equation*

$$u^n = Q(u, u', u'', \dots) \tag{2}$$

has a solution representable by generalized quadratures over K, it has a solution algebraic over K.

The logarithmic derivative $u = y'/y$ of any solution of the equation (1) satisfies the *associated with (1) generalized Riccati's equation of order* $(n-1)$, which is a particular case of the equation (2).

Rosenlicht showed that Theorem 1 easily follows from Theorem 4 applied to the corresponding generalized Riccati's equation.

The Rosenlicht's proof of Theorem 4 is not elementary: it makes use of the valuation theory for abstract differential fields.

Joseph Liouville in 1839 proved Theorem 1 for $n = 2$. Joseph Fels Ritt in 1948 simplified his proof. The logarithmic derivative $u = y'/y$ of any solution y of the homogeneous linear differential equation (1) of second order satisfies the Riccati's equation

$$u' + a_1 u + a_2 + u^2 = 0. \qquad (3)$$

To prove Theorem 1 for $n = 2$ J. Liouville and J.F. Ritt proved first Theorem 4 for the Riccati's equation (3). To do that J.F. Ritt considered a special one parametric family of solutions of (3) and used an expansion of these solutions as functions of the parameter into converging Puiseux series. J.F. Ritt used a generalization of the following theorem based on ideas suggested by Newton.

Consider an algebraic function $z(y)$ defined by an equation $P(y, z) = 0$ where P is a polynomial with coefficients in a subfield K of $\mathbb{C}$. Then all branches of the algebraic function $z(y)$ at the point $y = \infty$ can be developed into converging Puiseux series whose coefficients belong to a finite extension of the field K.

A *generalized Newton's Theorem* claims that the similar result holds if instead of a numerical field of coefficient one takes a field K whose element are meromorphic functions on a connected Riemann surface.

In the J.F. Ritt's book [1] this result is proved in the same way as its classical version using the Newton's polygon method.

In [7], [8] I found a proof of Theorem 4 which does not rely on the valuation theory.

It generalizes J.F. Ritt's arguments and provides an elementary proof of the classical Theorem 1.

The idea of the proof goes back to Liouville and J.F. Ritt. I came up with it trying to understand and comment the classical book [1] written by J.F. Ritt.

II. SPECIAL SOLUTIONS AND CLASSICAL APPROACH

Let us discuss different approaches to the problem (which are very close to each other) on a simple example.

Let K be some field of meromorphic functions on a connected domain on complex line which is closed under differentiation.

The second order homogeneous equation

$$y'' + ay' + by = 0 \tag{4}$$

over K, i.e. $a = a(x)$ and $b = b(x)$ are some functions from K.

Logarithmic derivative $u = y'/y$ of nonzero solution y of (4) satisfies the Riccati equation

$$u' + au + b + u^2 = 0. \tag{5}$$

Thus the problems of solving by quadratures equations (4) and (5) are equivalent.

Assume that we can not find any solution the Riccati equation (5) in the differential field K.

Can addition of the exponent $y = \exp x$ (satisfying the relation $y' = y$) to K help us to do that?

The answer is "No" if $\exp x$ is not an algebraic function over K, otherwise the corresponding algebraic extension could be helpful.

Let us explain why the answer is "No" by using modified Liouville's–Ritt's–Rosenlicht's arguments. After that we will comment on the original classical arguments for this example and on our modification.

There are two following possibilities:

1) the exponent y is algebraic over K.

In this case the extension by exponent could help. We consider such extension as an algebraic extension, not as an "honest" addition of the exponent.

2) the exponent y is transcendental over K.

Lemma 5 (Liouville's principle). *In the case 2) any element of $K\langle y\rangle$ can be represented in a unique way as a rational function in y with coefficients in K.*

The derivative R' of an element $R = R(y)$ is given by

$$R' = R'_x + R'_y y.$$

Proof. Indeed since

$$y' = y$$

any element in $K\langle y\rangle$ is representable as a rational function in y.

If some element a can be represented as a rational function in y in two different ways

$$a = R_1(y);\ a = R_2(y)$$

then

$$R_1(y) = R_2(y)$$

which means that y is algebraic over K which contradicts the assumption.

By the chain rule

$$\frac{d}{dx}R(y) = R'_x(y) + R'_y(y)y.$$

Thus all statements of the Liouville's principle for our example are proved. □

Theorem 6. *If the exponent y is transcendental over K and the Riccati equation (5) has a solution in $K\langle y\rangle$ then it has a solution in K.*

Proof. Assume that $u = R(y)$ satisfies (5). If we plug in the rational function R any solution $y(x)$ of the equation $y' = y$ then we still will get a solution of (5) under assumption that the plugging $y(x)$ in R is well defined.

Let us try to plug in R the solution $y \equiv 0$. If we succeed then we will have a solution $R(0)$ of the Riccati equation (5 in the field K.

But what if R has a pole of order $k > 0$ on the line $y \equiv 0$ in the (x, y) plane?

In that case the plugging $y(x) \equiv 0$ to R makes no sense.

From the formula for the derivative R' one can see that the derivative R' has a pole on this line $y \equiv 0$ of order not bigger than k.

Indeed the following Lemma holds:

Lemma 7. *If R is a rational function in y and it has a order m on the line $y \equiv 0$ then the function $R'_x + R'_y y$ has an order not smaller than m on this line.*

Proof. Let R be a function

$$y^m R_0$$

where R_0 can be restricted to $y \equiv 0$ as a nonzero function. Then

$$R' = my^m R_0 + y^{m+1}(R_0)'_y + y^m (R_0)'_x.$$

□

Thus if $u = R$ has a pole of order k then the term $u^2 = R^2$ in (5 has a pole of order $2k$ and all the terms u and u' in this Riccati equation have pole of order at most k. Thus if $u = R$ satisfies (5 then it can not have a pole on $y \equiv 0$.

Theorem is proven. □

Message:

It is useful to solve equations in the field of rational (or in algebraic) functions in many variables over a given differential field K.

One has more tools to deal with such solutions (for example the order of a rational function (or of a branch of algebraic function) on a hypersurface could be used.

Sometimes the problem of solving equation in rational (or in algebraic) functions) is related to the original problem.

Let us discuss on the above example the classical arguments and their similarity and difference with our approach.

In the classical arguments the line $y \equiv \infty$ plays the key role.

Instead of plugging $y \equiv 0$ into R one can plug $y \equiv \infty$. To do that one has to check first that R can not have a pole at $y \equiv \infty$.

As we saw above **it is enough to check that the order of $\frac{d}{dx}R(y)$ at infinity is not smaller than the order of $R(y)$ at infinity** (compare with Lemma 7).

In Liouville's–Ritt's arguments one considers the family

$$y_\mu = \mu \exp x$$

of solutions of the equation $y' = y$ where μ is an arbitrary constant. One develops the function $R(y_\mu)$ in Laurent (or in Puiseux) series as functions in the parameter μ about the point $\mu = \infty$. and differentiates these series as functions in x using the identity $y'_m u = y_\mu$.

Note that the line $y = \infty$ implicitly appears in these arguments: if μ approaches infinity then the function $y_\mu = \mu \exp x$ approaches $y = \infty$.

Rosenlich strongly simplified Liouville's–Ritt's arguments. Instead of developing R in converging power series about $y \equiv \infty$ he considered the order of R at $y \equiv \infty$. His considerations are analogous to Lemma 7. They provide the same result as in the Liouville's–Ritt's approach but in much simpler way.

After checking that R has no pole at $y \equiv \infty$ one completes arguments exactly as we did it above.

Thus all three approaches are very similar and follow the original brilliant Liouville's ideas.

One can ask what is so special about the line $y = \infty$, why it plays a so important role?

There is a good reason for that: it turns out that.

The "function $y \equiv \infty$" is a solution for the equation $y' = y$ and more generally it is a solution for the equation $y' = ay + b$, where $a, b \in K$.

This claim has the following meaning:
Let u be $1/y$, then

$$u' = -y'/y^2.$$

Thus if $y' = ay + b$ then

$$u' = (-ay - b)/y^2 = -au - bu^2.$$

The equation $u' = -au - bu^2$ has a special solution $u \equiv 0$.

It could be interpreted in the following way: the original equation $y' = ay + b$ has a special solution $y \equiv \infty$.

This observation explains the role of the line $y = \infty$ in classical arguments. It suggests that such special solutions play the key role in the problem. This suggestion is realized in our approach.

Remark 1. *As we saw above instead of solution $y = \exp x$ which is really important for us sometimes it is useful to consider degenerate solution $y \equiv 0$ (or $y \equiv \infty$) of the same equation $y' = y$.*

Similar effect in known in Schubert calculus: in order to find intersection number of degenerate cycles sometimes it is sufficient to find much more visual intersection number of degenerate cycles.

III. GENERALIZED RICCATI'S EQUATION

Let y be a a non zero element of some differential field and let u be its logarithmic derivative, i.e $y' = uy$.

Lemma 8. *1) There is a sequence*

$$D_0 = 1, \quad D_1 = u, \quad D_2 = u' + u^2, \ldots$$

of differential polynomials in u such that for $n = 0, 1, \ldots,$ the following relation

$$y^{(n)} = D_n(u)y \tag{6}$$

holds.

2) The sequence D_i satisfies the following recurrence relation

$$D_0 = 1; \quad D_{k+1} = \frac{dD_k}{dx} + uD_k. \tag{7}$$

3) The polynomial D_n has integral coefficients, its total degree in $u, \ldots, u^{(n-1)}$ is n, and its order is $n-1$.

4) Moreover D_n is a Rosenlicht type polynomial, i.e. its degree n homogeneous part equals to u^n.

All claims of Lemma 8 are straightforward.

Consider a homogeneous linear differential equation whose coefficients a_i belong to a differential field K

$$y^{(n)} + a_1 y^{(n-1)} + \cdots + a_n y = 0. \tag{8}$$

Definition 2. *The equation*

$$D_n + a_1 D_{n-1} + \cdots + a_n D_0 = 0 \tag{9}$$

of order $n-1$ is called the generalized Riccati's equation for the linear differential equation (8).

Lemma 9. *A non-zero element y satisfies the linear differential equation (8) if and only if its logarithmic derivative $u = y'/y$ satisfies the corresponding generalized Riccati's equation (9).*

Proof. For proving Lemma 9 in one direction one can divide (8) by y and use the follows from the identity

$$y^{(k)}/y = D_k(u).$$

□

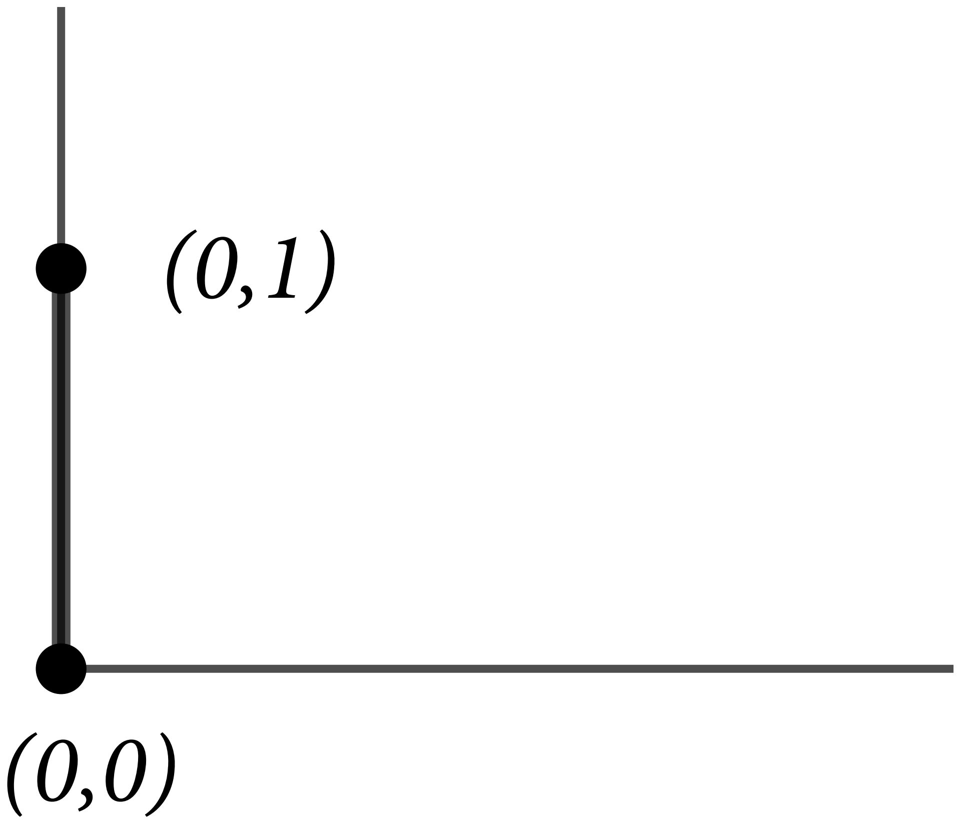
The generalized Riccati's equation satisfies the conditions of Rosenlicht Theorem (see Theorem 4). Thus Liouville's Theorem (see Theorem 1) follows from Rosenlicht Theorem via Lemma 9.

IV. DIFFERENTIAL POLYNOMIAL AND ITS NEWTON POLYTOPE

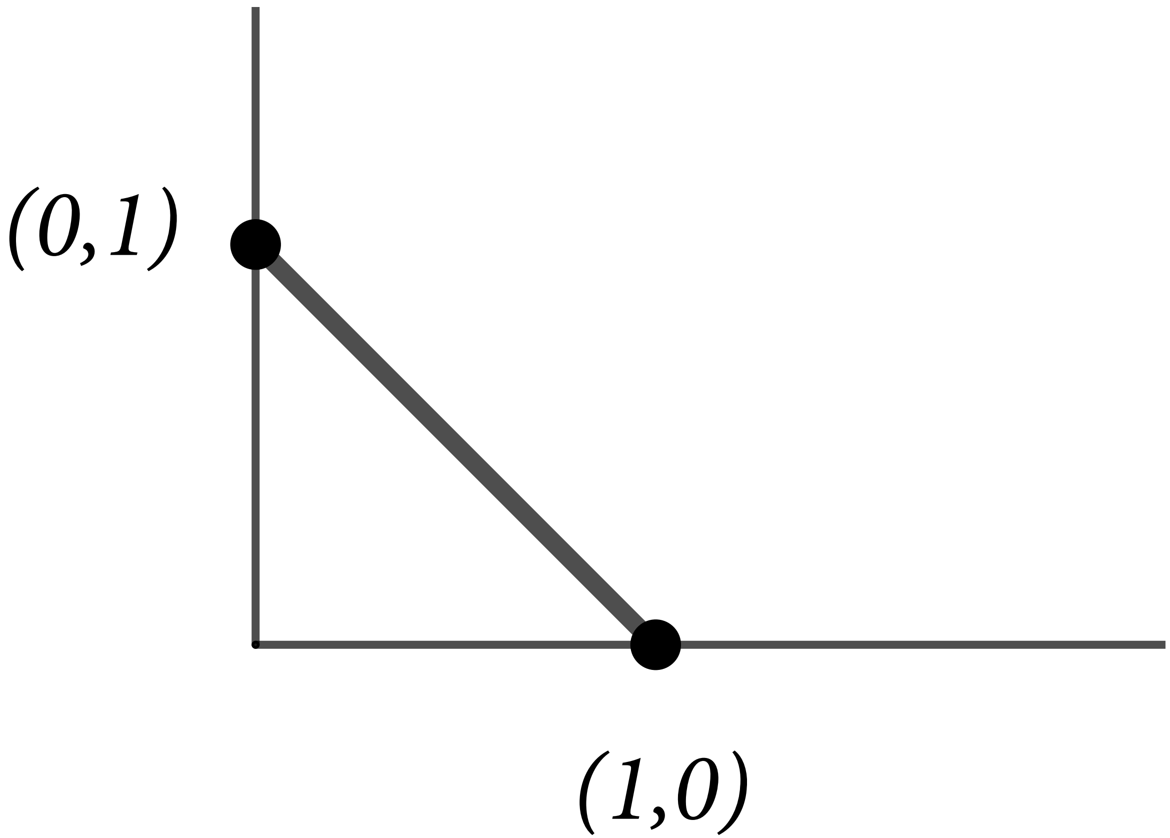
Let K be a differential field. A **differential polynomial over** K is a polynomial in y and its derivatives $y', \ldots, y^{(n-1)}$, where $n > 1$ over K.

One associates with a differential polynomial a usual polynomial over K in n ordered variables $x_1, \ldots, x_n$ where x_1 corresponds to y, x_2 corresponds to y' and so on.

Definition 3. *The Newton polytope $\Delta(P)$ of a differential polynomial $P(y, y', \ldots, y^{(n-1)})$ over K is the Newton polytope of the corresponding algebraic polynomial P over K in the ordered variables $x_1, \ldots, x_n$.*



Newton polytope of the differential polynomial $y' - a$



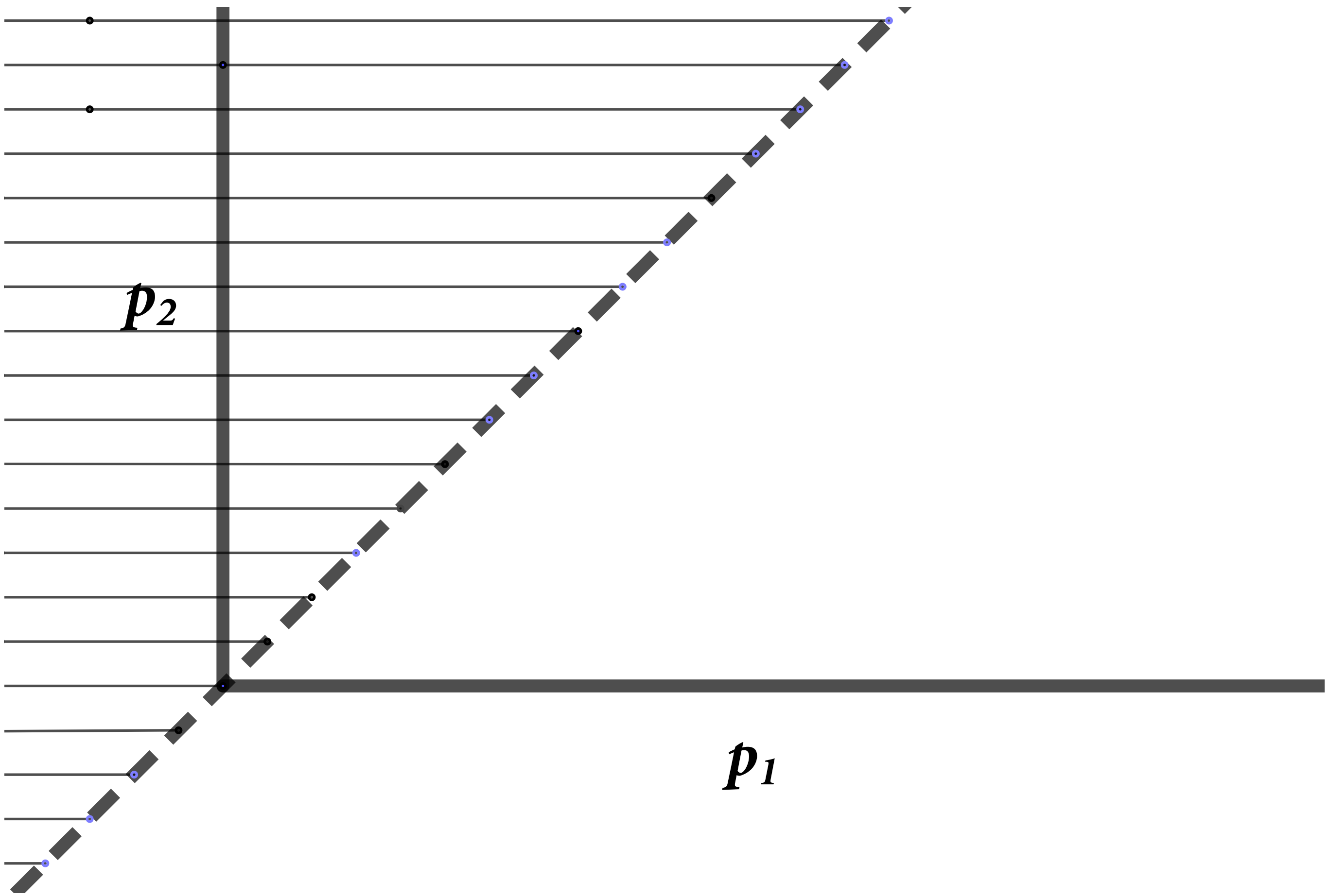
Newton polytope of the differential polynomial $y' - ay$

The polytope $\Delta(P)$ is contained in $\mathbb{R}^n$ with real coordinates $(m_1, \dots, m_n)$ whose integral points correspond to monomials $x_1^{m_1} \cdot \dots \cdot x_n^{m_n}$. In $\mathbb{R}^n$ we fix the standard inner product (such that $\langle e_i, e_j \rangle = \delta_{i,j}$ where $e_1, \dots, e_n$ is the standard bases in $\mathbb{R}^n$).

Definition 4. *An edge $E \subset \Delta(P)$ is* **horizontal** *if the inner product with the vector e_n is a constant k on E. Monomials belonging to such horizontal edge have the same degree k in x_n (corresponding to $y^{(n-1)}$).*

The inner product identifies $\mathbb{R}^n$ with $(\mathbb{R}^n)^*$. For pointing out that we consider a point in $\mathbb{R}^n$ as a covector we will denote its coordinates by $(p_1, \dots, p_n)$.

Definition 5. *A* **main cone** $C \subset (\mathbb{R}^n)^* = \mathbb{R}^n$ *is the cone defined by the following inequalities: $p_1 \leq p_2 \leq \dots \leq p_n$.*



The main cone $p_1 \leq p_2$ in $\mathbb{R}^2$

Definition 6. *The* **dual cone** $F^{\perp}$ **to a face** F *of a convex polytope* Δ *is the set of covectors* v *such that the linear function* $l(x) = \langle v, x \rangle$ *attains its minima on* Δ *exactly at the face* F*.*

The cone $F^{\perp}$ has dimension $n - \dim F$ and it is open in the relative topology of the $(n - \dim F)$-dimensional space containing $F^{\perp}$.

Definition 7. *An edge* $E \subset \Delta \subset \mathbb{R}^n$ *is* **compatible with the main cone** *if its dual cone* $E^{\perp}$ *has nonempty intersection with the main cone.*

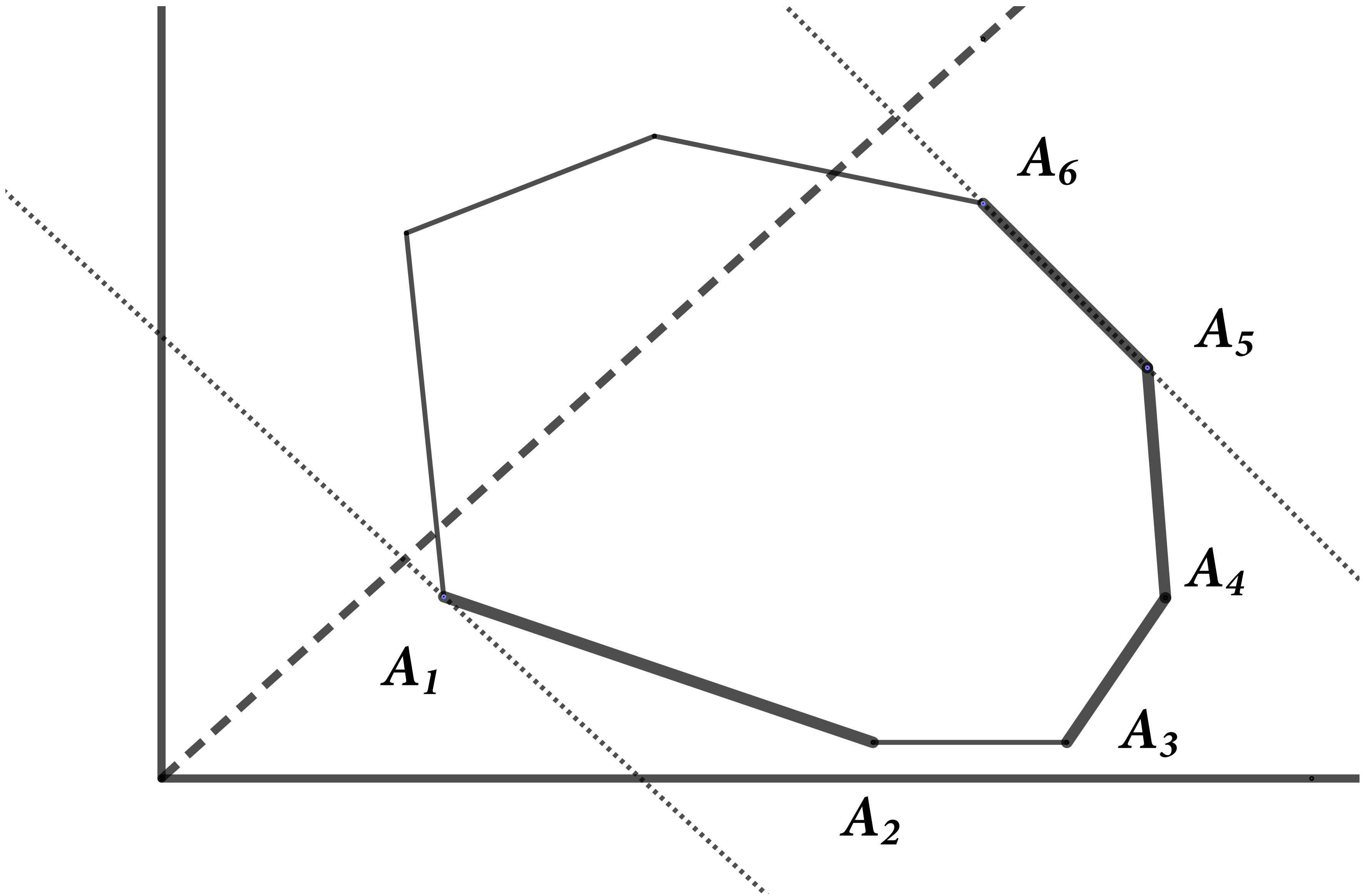
Let us consider the case $n = 2$. In this case the main cone $C \subset (\mathbb{R}^2)^* = \mathbb{R}^2$ is given by $p_1 \leq p_2$.

Example 1. *Let* $\Delta \subset \mathbb{R}^2$ *be a segment* E*. Then* $E^\perp$ *is the line orthogonal to* E*. It intersects* C *by a ray or by the line* $p_1 = p_2$*. Thus* E *is always compatible with cone. If* E *is not horizontal the main theorem (see below) is applicable. Thus it is applicable for the extension by integral and by exponent of integral.*

Example 2. *Let* $\Delta \subset \mathbb{R}^2$ *be a polygon. Its side* E *is compatible with* C *if and only if its inward-pointed normal* (p_1, p_2) *belongs to* C*, i.e.if* $p_1 \leq p_2$*. Thus the inward-pointed normal has to satisfy the following conditions:*

1) if $p_1 > 0$ *then* $p_2/p_1 \geq 1$;
2) if $p_1 = 0$ *then* $p_2 > 0$;
3) if $p_1 < 0$ *then* $p_2/p_1 \leq -1$.

An edge whose inward-pointed normal satisfies 2) is horizontal. The main theorem (see below) is not applicable to it.


A_6
A_5
A_4
A_1
A_3
A_2

V. DEFINITIONS

Definition 8. *A differential polynomial P over K is of* **Rosenlicht type** *if it has a unique monomial ay^N of degree $N = \deg P$. An equation $P(y, t', \ldots, y^{(M)}) = 0$ of arbitrary order M over K is of* **Rosenlicht type** *if P is a Rosenlicht type polynomial over K.*

Definition 9. *An extension $K_1 \supset K$ has* **an algebraic simplification property** *if the following condition holds:*

if there is a solution $y \in K_1$ of a Rosenlicht type equation $P = 0$ over K then there is an algebraic over K solution of $P = 0$.

Definition 10. *Let y be a transcendental over K element of a finitely generated field extension of K.* **Characteristic polynomial** *of y over K is differential polynomial P over the algebraic closure $\overline{K}$ of K of the smallest order and degree such that*

$$P(y, y', \dots) = 0.$$

The **polytope $\Delta(y)$ over** *K is the Newton polytope of P.*

Definition 11. *The polytope $\Delta(y) \subset \mathbb{R}^n$ over K* **allows degeneration** *if it contains an edge E which is not horizontal and which is compatible with the main cone $C \subset (\mathbb{R}^n)^* = \mathbb{R}^n$.*

Definition 12. *An extension* $K_1 \supset K$ **allows degeneration** *if* K_1 *is the algebraic closure of a differential field* $K\langle y\rangle$ *obtained by adding to* K *a transcendental over* K *element* y *and all its derivatives such that the polytope* $\Delta(y)$ *over* K *allows degeneration.*

Definition 13. *An extension* $F \supset K$ **allows a chain of degenerations** *if there is a chain of differential fields*

$$K = K_0 \subset K_1 \subset \cdots \subset K_k \supset F \tag{10}$$

such that for every $i = 1, \ldots, k$

or the extension $K_i \supset K_{i-1}$ *allows degeneration,*

or K_i *is the algebraic closure of* K_{i-1}*.*

VI. RESULTS

Theorem 10 (Main Theorem)**.** *If $K_1 \supset K$ allows degeneration then it has the algebraic simplification property for solutions of Rosenlicht type equations over K.*

The proof of Main Theorem is based on classical ideas due to Liouville, Ritt and Rozenlicht and on arguments from Newton polyhedra theory and toric geometry.

Theorem 11. *If $F \supset K$ allows a chain of degeneration then it has the algebraic simplification property for solutions of Rosenlicht type equations over K.*

Proof. Theorem 2 can be proven by induction using Theorem 1 as a step of induction. □

Theorem 12. *If a homogeneous linear differential equation over K has a nonzero solution in an extension $F \supset K$ which allows a chain of degenerations then the equation has a solution whose logarithmic derivative is algebraic over K.*

Proof. The logarithmic derivative of a nonzero solution of a homogeneous linear differential equation satisfies the corresponding generalized Riccaty equation which is a Rosenlicht type equation over K. Thus Theorem 3 follows from Theorem 2. □

Theorem 13. *If a homogeneous linear differential equation over K has a solution y in an extension $F \supset K$ which allows a chain of degenerations then y is representable in generalized quadratures over K.*

Proof. Using a nonzero solution of the homogeneous linear differential equation one can reduce the order of the equation. Using this procedure and the proof of Theorem 3 one can prove Theorem 4. □

References

[1] J.F. Ritt, Integration in Finite Terms, *Liouville's Theory of Elementary Methods*, Columbia University Press, New York, N. Y., 1948.

[2] M. Rosenlicht, *On the explicit solvability of certain transcendental equations*, Inst. Hautes Etudes Sci. Publ. Math. No. 36 (1969), 15–22.

[3] M. Rosenlicht, *An analogue of l'Hospital's rule*, Proc. Amer. Math. Soc. 37 (1973), 369–373.

[4] Michael F. Singer, *Solutions of linear differential equations in function fields of one variable*, Proc. Amer. Math. Soc. 54 (1976), 69–72.

[5] Michael F. Singer, *Differential Galois Theory.*

[6] A.G. Khovanskii, *Topological Galois Theory.*

[7] A.G. Khovanskii, *Solvability of equations by quadratures and Newton's theorem*, Arnold Mathematical Journal, V. 4, No 2, 193–211, 2018.

[8] A.G. Khovanskii, *Comments on J. F. Ritt's book "Integration in Finite Terms"*, arXiv:1908.02048 [math.AG] (Aug 2019).

[9] A.G. Khovanskii, *Newton polyhedra and good compactification theorem.*