Differential equations and Newton polyhedra

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Abstract. Our main theorem provides a new result on nonintegrability of linear differential equations in finite terms. Its statement uses geometry of convex polytopes. It is much stronger all known results of similar type. In particular it implies the classical criterium for solvability of such equations by quadratures. Its proof is based on classical ideas due to Liouville, Ritt and Rozenlicht and on arguments from Newton polyhedra theory and toric geometry.

Keywords. Integrability in finite terms, solvability by quadratures, linear differential equation, Newton polyhedron, toric geometry.

Introduction

Picard–Vessiot theorem (1910) provides a necessary and sufficient condition for solvability of linear differential equations of order n by generalized quadratures in terms of its Galois group (see [5], [6]). It is based on the differential Galois theory and is rather involved. J. Liouville in 1839 found an elementary criterium for such solvability for $n = 2$. J.F. Ritt simplified and clarified Liouville's arguments ([1], 1948). In 1973 M. Rosenlicht proved a similar criterium for arbitrary n. Rosenlicht arguments in many ways resemble Liouville's–Ritt's arguments, but he relies on the valuation theory and his proof is not elementary. (see [2], [3] and [4]).

Working on comments on Ritt's book I understood that the elementary Liouville's–Ritt's method based on developing solutions in Puiseux series as functions of a parameter works smoothly for arbitrary n and proves the same criterium (see [7], [8]).

Recently I returned back to this problem and found simple and powerful arguments which relate Liouville's–Ritt's approach and Rosenlicht's approach and strongly generalize them. Besides the classical ideas these arguments use Newton polyhedra theory and toric geometry.

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1. Differential polynomial and its Newton polytope

Let K be a differential field. A **differential polynomial over** K is a polynomial in y and its derivatives $y', \dots, y^{(n-1)}$, where $n \geq 1$ over K. One associates with a differential polynomial a usual polynomial over K in n ordered variables $x_1, \dots, x_n$ where x_1 corresponds to y, x_2 corresponds to y' and so on.

Definition 1. *The Newton polytope $\Delta(P)$ of a differential polynomial $P(y, y', \dots, y^{(n-1)})$ over K is the Newton polytope of the corresponding algebraic polynomial P over K in the ordered variables $x_1, \dots, x_n$.*

For example the Newton polytopes of the differential polynomials $P_1 = y' - b$ and $P_2 = y' - ay$ (where $a, b \in K$ and $a \neq 0$, $b \neq 0$) are correspondingly the segments OA and AB where $O = (0,0)$, $A = (0,1)$ and $B = (1,0)$.

The polytope $\Delta(P)$ is contained in $\mathbb{R}^n$ with real coordinates $(m_1, \dots, m_n)$ whose integral points correspond to monomials $x_1^{m_1} \cdot \dots \cdot x_n^{m_n}$. In $\mathbb{R}^n$ we fix the standard inner product (such that $\langle e_i, e_j \rangle = \delta_{i,j}$ where $e_1, \dots, e_n$ is the standard bases in $\mathbb{R}^n$).

Definition 2. *An edge $E \subset \Delta(P)$ is* **horizontal** *if the inner product with the vector e_n is a constant k on E. Monomials belonging to such horizontal edge have the same degree k in x_n (corresponding to $y^{(n-1)}$).*

The inner product identifies $\mathbb{R}^n$ with $(\mathbb{R}^n)^*$. For pointing out that we consider a point in $\mathbb{R}^n$ as a covector we will denote its coordinates by $(p_1, \dots, p_n)$.

Definition 3. *A* **main cone** *$C \subset (\mathbb{R}^n)^* = \mathbb{R}^n$ is the cone defined by the following inequalities: $p_1 \leq p_2 \leq \dots \leq p_n$.*

The dual cone $F^\perp$ to a face F of a convex polytope Δ is the set of covectors v such that the linear function $l(x) = \langle v, x \rangle$ attains its minima on Δ exactly at the face F. The cone $F^\perp$ has dimension $n - \dim F$ and it is open in the relative topology of the $(n - \dim F)$-dimensional space containing $F^\perp$.

Definition 4. *An edge $E \subset \Delta \subset \mathbb{R}^n$ is* **compatible with the main cone** *if its dual cone $E^\perp$ has nonempty intersection with the main cone.*

Let us consider the case $n = 2$. In this case the main cone $C \subset (\mathbb{R}^2)^* = \mathbb{R}^2$ is given by $p_1 \leq p_2$.

Example 1. *Let $\Delta \subset \mathbb{R}^2$ be a segment E. The dual cone $E^\perp$ is the line passing through the origin and orthogonal to E. It intersects C by a ray or by the line $p_1 = p_2$. Thus E is always compatible with the main cone. If E is not horizontal the main theorem (see below) is applicable. Thus it is applicable for the extension by integral $y' = b$ and for the extension by exponent of integral $y' = ay$.*

Example 2. *Let $\Delta \subset \mathbb{R}^2$ be a polygon A side E of the polygon Δ is compatible with the main cone if and only if its inward-pointed normal (p_1, p_2) belongs to the main cone C, i.e. $p_1 \leq p_2$. In other words the inward-pointed normal has to satisfy the* *following conditions: 1)* $p_1 > 0$ *then* $p_2/p_1 \geq 1$*; 2) if* $p_1 = 0$ *then* $p_2 > 0$*; 3) if* $p_1 < 0$ *then* $p_2/p_1 \leq -1$.

Note that an edge whose inward-pointed normal satisfies 2) is horizontal. The main theorem (see below) is not applicable to such edge.

2. Algebraic simplification of solutions and characteristic polynomial

Definition 5. *A differential polynomial* P *over* K *is of* **Rosenlicht type** *if it has a unique monomial of the highest degree* $N = \deg P$ *and this monomial is* ay^N*. An equation* $P(y, t', \dots, y^{(N)}) = 0$ *of arbitrary order*N *over* K *is of* **Rosenlicht type** *is* P *is a Rosenlicht type polynomial over* K.

Definition 6. *An extension* $K_1 \supset K$ *has* **an algebraic simplification property** *for solutions of Rosenlicht type equations if the following condition holds: if there is a solution* $y \in K_1$ *of a Rosenlicht type equation* $P = 0$ *over* K *then there is an algebraic over* K *solution of the same equation* $p = 0$.

Definition 7. *Let* y *be a transcendental over* K *element such that the a differential field* $K\langle y \rangle$ *is a finite field extension of* K*. A* **characteristic polynomial** *of* y *over* K *is an irreducible differential polynomial* P *over the algebraic closure* $\overline{K}$ *of* K *such that* $P(y, y', \dots) = 0$*. The* **polytope** $\Delta(y)$ **over** K *is the Newton polytope of the characteristic polynomial of* y *over* K.

Definition 8. *The polytope* $\Delta(y) \subset \mathbb{R}^n$ *over* K **allows degeneration** *if it contains an edge* E *which is not horizontal and which is compatible with the main cone* $C \subset (\mathbb{R}^n)^* = \mathbb{R}^n$.

Definition 9. *An extension* $K_1 \supset K$ **allows degeneration** *if* K_1 *is the algebraic closure of a differential field* $K\langle y \rangle$ *obtained by adding to* K *a transcendental over* K *element* y *and all its derivatives such that the polytope* $\Delta(y)$ *over* K *allows degeneration.*

An extension $F \supset K$ **allows a chain of degenerations** *if there is a chain of differential fields*

$$K = K_0 \subset K_1 \subset \dots \subset K_k \tag{1}$$

such that for every $i = 1, \dots, k$ *the extension* $K_i \supset K_{i-1}$ *allows degeneration and* K_k *contains* F.

3. Results

Theorem 1 (Main Theorem). *If* $K_1 \supset K$ *allows degeneration then it has the algebraic simplification property for solutions of Rosenlicht type equations over* K.

The proof of Main Theorem is based on classical ideas due to Liouville, Ritt and Rozenlicht and on arguments from Newton polyhedra theory and toric geometry.

Theorem 2. *If $F \supset K$ allows a chain of degeneration then it has the algebraic simplification property for solutions of Rosenlicht type equations over K.*

Proof. Theorem 2 can be proven by induction using Theorem 1 as a step of induction. □

Theorem 3. *If a homogeneous linear differential equation over K has a nonzero solution in an extension $F \supset K$ which allows a chain of degenerations then the equation has a solution whose logarithmic derivative is algebraic over K.*

Proof. The logarithmic derivative of a nonzero solution of a homogeneous linear differential equation satisfies the corresponding generalized Riccaty equation (see for example [7]) which is a Rosenlicht type equation over K. Thus Theorem 3 follows from Theorem 2. □

Theorem 4. *If a homogeneous linear differential equation over K has a solution y in an extension $F \supset K$ which allows a chain of degenerations then y is representable in generalized quadratures over K.*

Proof. Using a solution of the homogeneous linear differential equation one can reduce the order of the equation. Using this procedure and the proof of Theorem 3 one can prove Theorem 4. □

References

[1] J.F. Ritt, Integration in Finite Terms, *Liouville's Theory of Elementary Methods*, Columbia University Press, New York, N. Y., 1948.

[2] M. Rosenlicht, *On the explicit solvability of certain transcendental equations*, Inst. Hautes Etudes Sci. Publ. Math. No. 36 (1969), 15–22.

[3] M. Rosenlicht, *An analogue of l'Hospital's rule*, Proc. Amer. Math. Soc. 37 (1973), 369–373.

[4] Michael F. Singer, *Solutions of linear differential equations in function fields of one variable*, Proc. Amer. Math. Soc. 54 (1976), 69–72.

[5] Michael F. Singer, *Differential Galois Theory.*

[6] A.G. Khovanskii, *Topological Galois Theory.*

[7] A.G. Khovanskii, *Solvability of equations by quadratures and Newton's theorem*, Arnold Mathematical Journal, V. 4, No 2, 193–211, 2018.

[8] A.G. Khovanskii, *Comments on J. F. Ritt's book "Integration in Finite Terms"*, arXiv:1908.02048 [math.AG] (Aug 2019).

[9] A.G. Khovanskii, *Newton polyhedra and good compactification theorem.*

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