Algorithms and computer algebra software for solving polynomial equation in one or two variables

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Abstract. Here we demonstrate two new methods of solution of polynomial equations, based on constructing a convex polygon, and provide description of corresponding software. The first method allows to find approximate roots of a polynomial by means of the Hadamard polygon. The second one allows to compute branches of an algebraic curve near its singular point and near infinity by means of the Newton polygon and to draw sketches of real algebraic curves in the plane. Computer algebra algorithms are specified, which allow to investigate any complex cases.

Introduction

Here we present two new methods for solving polynomial equations based on the construction of a convex polygon from a polynomial. The first method allows one to find approximate roots of a polynomial using the Hadamard broken line (section 2). The second method allows one to find branches of an algebraic curve near its singular point and near infinity with the Newton polygon (section 3). It also allows one to construct sketches of real algebraic curves in the plane. These methods can be generalized to higher dimensions [1].

All algorithms are provided with descriptions of their software implementation in various computer algebra systems.

New points of this work are the following:

- the concept of the cone of the problem is actively used ;
- the application of Newton's polygon to find branches of a curve near infinity is given;
- the theory of Hadamard's broken line method is given;
- computer algebra software is discussed for all algorithms.

For extended version of this work with many examples, listings of program for computer algebra systems `Maple`, `Sympy`, `Mathematica` and for detailed references see [2].

1. Polyhedron and normal cone

Let in $\mathbb{R}^n$ given several points $\{Q_1, \ldots, Q_k\} = \mathbf{S}$. Their convex hull

$$\Gamma(\mathbf{S}) = \left\{ Q = \sum_{i=1}^{k} \mu_i Q_i, \mu_i \geqslant 0, \sum \mu_i = 1 \right\}$$

is a polyhedron. Its boundary $\partial\Gamma$ consists of vertices $\Gamma_j^{(0)}$, edges $\Gamma_j^{(1)}$ and faces $\Gamma_j^{(d)}$ of different dimensions $d : 1 < d \leqslant n-1$. If the real n-vector $P = (p_1, \ldots, p_n)$ is given, then the maximum and minimum of the scalar product $\langle P, Q \rangle = p_1 q_1 + \ldots + p_n q_n$ on $\mathbf{S}$ are reached at points Q_i that lie on the boundary $\partial\Gamma$. For each boundary element $\Gamma_j^{(d)}$ (including vertices $\Gamma_j^{(0)}$ and edges $\Gamma_j^{(1)}$), we identify the set of vectors P whose maximum $\langle P, Q \rangle$ is reached on points $Q_i \in \Gamma_j^{(d)}$. This will be its *normal cone*

$$\mathbf{U}_j^{(d)} = \left\{ P : \langle P, Q' \rangle = \langle P, Q'' \rangle > \langle P, Q'' \rangle \text{ for } \ Q', Q'' \in \Gamma_j^{(d)}, Q'' \in \Gamma \backslash \Gamma_j^{(d)} \right\}.$$

The vector P lies in the space $\mathbb{R}_*^n$, dual to the space $\mathbb{R}^n$.

Let us be interested not in the whole boundary $\partial\Gamma$, but only a part of it corresponding to some set $\mathcal{K}$ of directions P. Then let us call the set $\mathcal{K}$ the *cone of the problem.* It is not necessarily convex. By $\partial\Gamma(\mathcal{K})$ we denote the part of the boundary $\partial\Gamma$ for whose elements $\Gamma_j^{(d)}$ their normal cones $\mathbf{U}_j^{(d)}$ intersect with the cone of problem $\mathcal{K}$. Let us call $\partial\Gamma(\mathcal{K})$ — *boundary of the problem.*

1.1. Software for convex hull and normal cones computation

Various software is available for working with convex sets. Here we briefly describe only those programs that can be used both to compute convex hulls and to compute their normal cones.

The `Qhull` package is used in many application software packages, both commercial and free, the `Qhull` package has a software interface with the `Matlab` scientific calculation system, `GNU Octave`, computer algebra systems `Mathematica` and `Maple`, libraries `SciPy` and `geometry` for programming languages `Python` and `R` respectively. The main feature of the package is that the calculations are performed using real numbers rather than in the field of rational numbers, which is convenient when working with the Hadamard polyhedron. When calculating the Newton polyhedron, additional steps are required to bring the results of the calculations to rational values.

Since the 2015 version, the `Maple` computer algebra system includes the `PolyhedralSets` package. It allows, in particular, to compute the convex hull of a set, to give its H- or V-representations, i.e., either as equations of hyperplanes of the boundary, or as a set of extreme points and rays, the linear combination of which gives an unbounded convex hull. In this package all calculations are performed in the field of rational numbers, which somewhat simplifies its use for the study of the Newton polyhedron, but makes it useless when working with the

Hadamard polyhedron. Note that `PolyhedralSets` has extremely low performance compared to `Qhull`.

2. Hadamard broken line method

Let us describe a new method for computing approximate values of roots of the polynomial

$$f_m(x) = \sum_{k=0}^{m} a_k x^k. \tag{1}$$

To do this, the points in the real plane q_1, q_2 are plotted $\check{Q}_k = (q_1, q_2) = (k, \ln |a_k|)$, $k = 0, \ldots, m$, forming the *supersupport* $\check{\mathbf{S}} = \{\check{Q}_0, \ldots, \check{Q}_m\}$, and their convex hull is constructed $\Gamma(\check{\mathbf{S}}) = \left\{\check{Q} = \sum\limits_{k=0}^{m} \mu_k \check{Q}_k, \mu_k \geqslant 0, \sum_{k=0}^{m} \mu_k = 1\right\} \stackrel{\text{def}}{=} \mathbf{H}(f_m)$, which is called *Hadamard's polygon* [3] (Hadamard, 1893). The boundary $\partial\mathbf{H}$ is a broken line. Each edge $\Gamma_j^{(1)}$ and vertex $\Gamma_j^{(0)}$ of this boundary $\partial\mathbf{H}$ corresponds to a boundary subset $\mathbf{S}_j^{(d)}$ of points $\check{Q}_k$ lying on $\Gamma_j^{(d)}$, and the truncated polynomial

$$\check{f}_j^{(d)}(x) = \sum a_k x^k \text{ by } \check{Q}_k \in \mathbf{S}_j^{(d)}. \tag{2}$$

If $\Gamma_j^{(d)}$ is a vertex ($d = 0$), then the truncated polynomial (2) is a monomial that has no nonzero root. If $\Gamma_j^{(d)}$ is an edge ($d = 1$), then the truncated polynomial (2) has nonzero roots, which give approximate values for the roots of the full polynomial (1). Except exceptional cases, the truncated polynomials (2) are significantly simpler than the original polynomial (1), and their roots are easier to compute.

Since the vector $(p_1, 1)$ lies in the upper half-plane of the dual plane $\mathbb{R}_*^2$, the cone of the problem here $\mathcal{K} = \{P = (p_1, p_2) : p_2 > 0\}$, i.e. — this is the upper half-plane. It corresponds to the upper part of the boundary $\partial\mathbf{H}$. It will be called *Hadamard's broken line* and denoted by $\widetilde{\mathbf{H}}$.

Examples with successful application of the Hadamard broken line method see in [1, 2].

Example 1. Using the Hadamard broken line, let's find the approximate values of the roots of the polynomial $f(x) = 3+3x^2+x^4$ with roots $x_k^0 = \pm\frac{1}{2}\sqrt{-6 \pm 2i\sqrt{3}} \approx \pm 0.340625 \pm 1.271230i$. Its Hadamard broken line is stretched over three vertices $(0, \ln 3)$, $(2, \ln 3)$, $(4, 0)$.

It has two edges $\Gamma_1^{(1)}$ and $\Gamma_2^{(2)}$, which correspond to the truncated polynomials $\check{f}_1^{(1)}(x) = 3 + 3x^2$ and $\check{f}_2^{(1)}(x) = 3x^2 + x^4$, with corresponding roots $x_{1,2} = \pm i$ and $x_{3,4} = \pm\sqrt{3} \approx \pm 1.732051i$.

Iterations of these roots by Newton's method do not converge to exact values of the roots. For $x_{1,2}$ we obtain a periodic sequence of values with period 2 near $\pm 1.042i$. Root iterations of $x_{3,4}$ generally exhibit chaotic behavior. ■

In general, the Hadamard broken line method consists in replacing the original polynomial by a set of simpler polynomials whose roots are computed easily and give such approximations of the roots of the original polynomial that are good enough for numerical refinement by Newton's method.

Let us find the reason for the distance between the exact roots of x_i^0 of the polynomial (1) and the approximate roots of x_i obtained by the Hadamard broken line method. Assume

$$\alpha_k = \ln|a_k| \text{ and } \beta_k = \begin{cases} a_k/|a_k|, & \text{if } a_k \neq 0, \\ 0, & \text{if } a_k = 0, \end{cases}$$

where $k = 0, \dots, m$. Then the polynomial (1) can be written as $f(x) = \sum\limits_{k=0}^{m} \beta_k x^k e^{\alpha_k}$, where all $|\beta_k| = 1$ and 0. Let's match it with the sum of $g(x,y) = \sum \beta_k x^k y^{\alpha_k}$ and consider the curve $\mathcal{F}$ defined by Equation

$$g(x,y) = 0. \tag{3}$$

At $y \to \infty$ the branches of the curve $\mathcal{F}$ are approximated by solutions of the shortened equations $\hat{g}_l^{(1)}(x,y) = 0$, corresponding to the edges of $\widetilde{\Gamma}_l^{(1)}$ of the Hadamard broken line $\widetilde{\mathbf{H}}$ of the polynomial (1). If in the interval

$$y \in [e, \infty] \tag{4}$$

these branches do not intersect or merge, then the approximate roots of x_i obtained with the Hadamard broken line are close to the exact roots of x_i^0 of the polynomial (1). But if in the interval (4) these branches intersect or merge, then at $y = e$ their mutual placement differs from the limiting location at $y \to \infty$. But the branches of the curve (3) can intersect or merge only at points $\left(x^0, y^0\right)$: $g(x^0, y^0) = \dfrac{\partial g}{\partial x}(x^0, y^0) = 0$, i.e., where the discriminant Δ of the polynomial g from x is zero.

Therefore, if the curve (3) has no points

$$\left(x^0, y^0\right) \text{ with } \Delta\left(x^0, y^0\right) = 0 \tag{5}$$

in the interval (4), then the approximate roots x_i are close to the exact roots x_i^0 of the polynomial (1). But if the curve (3) has (4) points (5) in the interval, then the approximate roots x_i may be far from the exact roots x_i^0 of the polynomial (1). For the proximity of real roots of x_i and x_i^0, the absence of real non-isolated points (5) in the interval (4) is sufficient.

3. Plane algebraic curve

Let $f(x_1, x_2)$ be a polynomial with real or complex coefficients. The set of solutions x_1, x_2 of the equation

$$f(x_1, x_2) = 0 \tag{6}$$

in $X = (x_1, x_2) \in \mathbb{R}^2$ or $\mathbb{C}^2$ is called a *plane algebraic curve $\mathcal{F}$*.

A point $X = X^0$, $f(X^0) = 0$ is called a *simple* point of a curve $\mathcal{F}$ if the vector $(\partial f/\partial x_1, \partial f/\partial x_2)$ in it is nonzero. Otherwise, the point X^0 is called *singular*. By a shift, move the point X^0 to the origin of coordinates.

3.1. Local analysis of a simple point

Theorem 1 (Cauchy). *If at $X^0 = 0$ the derivative $\partial f/\partial x_i \neq 0$, then all solutions to the equation* (6) *near point $X^0 = 0$ are contained in the expansion*

$$x_i = \sum_{k=1}^{\infty} b_k x_j^k, \tag{7}$$

where b_k — constants, and $j = 2 - i$.

3.2. Local analysis of the singular point $X^0 = 0$ and points at infinity

Let's write the polynomial $f(X)$ as

$$f(X) = \sum f_Q X^Q \text{ by } Q \geqslant 0, \quad Q \in \mathbb{Z}^2, \tag{8}$$

where $X = (x_1, x_2)$, $Q = (q_1, q_2)$, $X^Q = x_1^{q_1} x_2^{q_2}$, $f_Q \in \mathbb{C}$ are constants. Let $\mathbf{S}(f) = \{Q : f_Q \neq 0\} \subset \mathbb{R}^2$. The set $\mathbf{S}$ is called the *support* of the polynomial $f(X)$. Let it consist of points $Q_1, \ldots, Q_k$. The convex hull of the support $\mathbf{S}(f)$ is the set

$$\Gamma(\mathbf{S}) = \left\{ Q = \sum_{j=1}^{k} \mu_j Q_j,\ \mu_j \geqslant 0,\ \sum_{j=1}^{k} \mu_j = 1 \right\} \stackrel{\text{def}}{=} \mathbf{N}(f),$$

which is called *Newton's polygon*. The boundary $\partial\mathbf{N}(f)$ consists of vertices $\Gamma_j^{(0)}$ and edges $\Gamma_j^{(1)}$, where j —is the number.

Each generalized face $\Gamma_j^{(d)}$ corresponds to: its *boundary subset* $\mathbf{S}_j^{(d)} = \mathbf{S} \cap \Gamma_j^{(d)}$, its a *truncated* polynomial $\hat{f}_j^{(d)}(X) = \sum f_Q X^Q$ by $Q \in \mathbf{S}_j^{(d)}$ and its *normal cone* $\mathbf{U}_j^{(d)} = \{P : \langle P, Q' \rangle = \langle P, Q'' \rangle > \langle P, Q'' \rangle, Q'' \in \Gamma_j^{(d)}, Q'' \in \Gamma \backslash \Gamma_j^{(d)}\}$, where $P = (p_1, p_2) \in \mathbb{R}_*^2$, and the plane $\mathbb{R}_*^2$ is conjugate to the plane $\mathbb{R}^2$.

Let $x_1 \in \mathbb{C}$, $x_1 \to 0$ or ∞, and $o(1)$ be a function of x_1, which tends to zero at that. On the curve $x_2 = b x_1^p (1 + o(1))$, where $b = \text{const} \in \mathbb{C}$, $p \in \mathbb{R}$, monomial

$$f_Q X^Q = f_Q b^{q_2} \left\{ \exp\left[\ \langle Q, P \rangle\, \omega |\ln x_1|\ \right] \right\} (1 + o(1)). \tag{9}$$

In this case $P = (1, p)$, and $\omega \stackrel{\text{def}}{=} \operatorname{sgn} \ln |x_1| = \begin{cases} -1, & \text{if} \quad x_1 \to 0, \\ \ \ 1, & \text{if} \quad x_1 \to \infty \end{cases}$. This means that for given P and ω the largest moduli have those monomials (9) of the sum (8) on which the value

$$\omega \langle Q, P \rangle \ \text{ by } Q \in \mathbf{S} \tag{10}$$

reaches a maximum.

If $x_1 \to 0$, $\omega = -1$ and the vector $\omega P = (-1, -p)$. Hence, here the cone of the problem $\mathcal{K}$ is the left half-plane of the plane $\mathbb{R}_*^2$ and the points Q with maximal values of the product (10) lie on the left side of the boundary $\partial\mathbf{N}$.

If $x_1 \to \infty$, $\omega = 1$ and vector $\omega P = (1, p)$. Hence, here the cone of the problem $\mathcal{K}$ is the right half-plane of the plane $\mathbb{R}^2_*$ and the points Q with the largest values of the product (10) lie on the right side of the boundary $\partial\mathbf{N}$.

We will look for solutions to the equation

$$f(X) \stackrel{\text{def}}{=} \sum_{Q \in \mathbf{S}} f_Q X^Q = 0 \qquad (11)$$

In the form of expansion

$$x_2 = b_1 x_1^{p_1} + b_2 x_1^{p_2} + b_3 x_1^{p_3} + \cdots, \qquad (12)$$

where f_Q, $b_k = \text{const} \in \mathbb{C}$, $Q \in \mathbb{R}^2$, $p_k = \text{const} \in \mathbb{R}$, $\omega p_k > \omega p_{k+1}$.

In these expansions, the exponents of degree p_k increase with k if $x_1 \to 0$, and decrease if $x_1 \to \infty$.

So, it is proved that

Theorem 2. *For solutions* (12) *of equation* (11)*, the truncated solution* $x_2 = b_1 x_1^{p_1}$ *is the solution to the truncated equation*

$$\hat{f}_j^{(d)}(X) = 0, \qquad (13)$$

corresponding to the boundary element $\Gamma_j^{(d)}$ *with the external normal vector* $\omega(1, p_1) \in \mathbf{U}_j^{(d)}$.

The truncated equation (13) uniquely determines the sign of ω and the index of degree p_1. If in the sum (8) all vector indexes of degree $Q = (q_1, q_2)$ have rational components q_1 and q_2, then the index p_1 is–rational. For the coefficient b_1 we obtain the algebraic equation $\hat{f}_j^{(1)}(1, b_1) = 0$.

Theorem 3. *For the polynomial equation* (11)*, all solutions* $x_2(x_1)$ *are expanded into a series of the form* (12)*, where all exponents of degree* p_k *are rational numbers with a common denominator.*

For the neighborhood of the point $X = 0$, the theorem 3 is that of V. Puiseau, 1850, i.e., for the cone problem $\mathcal{K} = \{P = (p_1, p_2) : p_1, p_2 < 0\}$. The corresponding part (lower left) of the boundary $\partial\mathbf{N}$ is called Newton's *broken line.* The expansions of the theorem 1 converge, so all expansions (12) for solutions of polynomial equations (11) also converge.

3.3. Real curve

If the polynomial (6) has all coefficients of f_Q real, then the equation (8) at $x \in \mathbb{R}$ defines a real curve F on the plane x_1, x_2. For it in the expansion (7) of Theorem 1 all the coefficients b_k are real, and from the expansion (12) of Theorem 3 only those in which all coefficients b_k are real have sense. Since they are computed sequentially, at each step only real coefficients of b_k should be left, and the complex coefficients should be discarded.

At the singular point $X = 0$ the expansion (7) starts with the quadratic terms $f = g_{20}x_1^2 + g_{11}x_1x_2 + g_{02}x_2^2 + \ldots \stackrel{\text{def}}{=} g_2(x_1, x_2) + \ldots$

Theorem 4. *If the discriminant* $\Delta = g_{11}^2 - 4g_{20}g_{02}$ *of the quadratic form* g_2 *is less than zero, then the curve* F *has no real branches passing through the critical point* $X = 0$.

Let there be two polynomials of degree 2 and 3, respectively:

$$g = g_0x^2 + g_1x + g_2,\ h = h_0x^3 + h_1x^2 + h_2x + h_3.$$

If polynomials g and h have a common root, then the resultant $\mathrm{Res}(g, h)$ is identically zero.

Theorem 5. *Let at the singular point* $X = 0$

$$f = f_2(X) + f_3(X) + \dots,$$

where $f_k(X)$ *— are homogeneous forms of degree* k *by* x_1, x_2*, the form* $f_2 \not\equiv 0$ *and its discriminant* $\Delta = 0$*, and the resultant* $\mathrm{Res}(f_2, f_3) \neq 0$*, then near point* $X = 0$ *the curve* F *— is a semicubic parabola of the form*

$$x_2 = \alpha x_1 \pm \beta x_1^{3/2} + \dots,$$

where α *is a multiple root of the polynomial* $f_2(1, \alpha)$.

3.4. Sketch of a real curve

Let the polynomial $f(X)$ have all coefficients real, then on the real plane $X \in \mathbb{R}^2$ we can draw all its branches using the local analysis described above. Let us divide this procedure into several steps.

Step 1. We expand the polynomial $f(X)$ into polynomial multipliers using various computer algebra algorithms. Then we construct curve sketches for each irreducible factor $f_l(X)$ separately.

Step 2. Find all real finite singular points of X^0 of the curve $f = 0$ in which

$$f(X^0) = 0,\quad \frac{\partial f}{\partial x_1}(X^0) = 0,\quad \frac{\partial f}{\partial x_2}(X^0) = 0,$$

using the exclusion method or the Gröbner basis method.

Step 3. Near each singular point X^0, find all real branches of the form (12) by moving it to the origin and using methods of subsections 3.2 and 3.3 .

Step 4. Find the intersection points of the curve with the axes $x_1 = 0$ and $x_2 = 0$ as solutions of the equations $f(0, x_2) = 0$ and $f(x_1, 0) = 0$, using the methods of section 2 and refining them by the Theorem 1.

Step 5. Find the finite intersection points of the curve with infinities $x_1 = \infty$ and $x_2 = \infty$ by truncated equations with $P = (1, 0)$ and $P = (0, 1)$. For each of these, compute the initial terms of the (12) type expansion.

Step 6. Find the branches of the curve at $x_1 \to 0$, $x_2 \to \infty$, using part of the $\partial\mathbf{N}$ boundary with the cone problem $\mathcal{K}_1 = \{P : p_1 < 0, p_2 > 0\}$. Similarly, when $x_1 \to \infty$, $x_2 \to 0$ — with the problem cone $\mathcal{K}_2 = \{P : p_1 > 0, p_2 < 0\}$.

Step 7. Find the branches of the curve at $x_1, x_2 \to \infty$ using part of the boundary $\partial\mathbf{N}$ with the cone problem $\mathcal{K}_3 = \{P : p_1 > 0, p_2 > 0\}$.

Step 8. Connect the found pieces of curve branches, given that outside the special points of X^0 the curve branches do not intersect.

3.5. Software for plane curve investigation

The `Maple` system has an excellent package `algcurves` that allows to study planar algebraic curves: build their sketches with high precision, calculate their genus, find singular points, for curves of genus 0 find rational parameterization, for elliptic curves bring to Weierstrass normal form. The package allows to construct a sketch of the curve $f(x, y) = 0$ by numerical integration of the differential equation $f'_x + f'_y y' = 0$ for some set of initial conditions defined by points in which at least one of the partial derivatives of the function $f(x, y)$ is equal to zero.

Since version 12, `Wolfram Mathematica` has included an `AsymptoticSolve` procedure that implements an asymptotic representation of solutions to equations or systems of equations (not necessarily algebraic) in the form of either Taylor, Laurent, or Puiseau series near finite or infinite points. If the point is singular, the procedure tries to calculate the asymptotic expansions of all branches. In this case we can specify that we should restrict ourselves to real expansions only.

References

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