

Algorithms and computer algebra software for solving polynomial equation in one or two variables

Alexander D. Bruno, Alexander B. Batkhin
abruno@keldysh.ru
batkhin@gmail.com

Keldysh Institute of Applied Mathematics of RAS

PCA-2021, April 19–23, 2021

Talk outlook

1. Polyhedron and normal cone

1.1 Software for convex hull and normal cones computation

2. Hadamard broken line method

2.1 Hadamard broken line theory

2.2 Some software for root computation

3. Plane algebraic curve

3.1 Local analysis of a simple point

3.2 Local analysis of the singular point $X^0 = 0$ and points at infinity

3.3 Real curve

3.4 Sketch of a real curve

3.5 Software for plane curve investigation

3.6 Big example

Abstract

Here we demonstrate two new methods of solution of polynomial equations, based on constructing a convex polygon, and provide description of corresponding software. The first method allows to find approximate roots of a polynomial by means of the Hadamard polygon. The second one allows to compute branches of an algebraic curve near its singular point and near infinity by means of the Newton polygon and to draw sketches of real algebraic curves in the plane. Computer algebra algorithms are specified, which allow to investigate any complex cases.

In a more detailed form this material is published in the preprint

Bruno A. D., Batkhin A. B. Introduction to nonlinear analysis of algebraic equations. // KIAM Preprints. 2020. No. 87. URL: https://keldysh.ru/papers/2020/prep2020_87.pdf ; (in Russian),

which is in the public domain.

1. Polyhedron and normal cone

1.1 Software for convex hull and normal cones computation

2. Hadamard broken line method

2.1 Hadamard broken line theory

2.2 Some software for root computation

3. Plane algebraic curve

3.1 Local analysis of a simple point

3.2 Local analysis of the singular point $X^0 = 0$ and points at infinity

3.3 Real curve

3.4 Sketch of a real curve

3.5 Software for plane curve investigation

3.6 Big example

1. Polyhedron and normal cone (1)

Let in \mathbb{R}^n given several points $\{Q_1, \dots, Q_k\} = \mathbf{S}$. Their convex hull

$$\Gamma(\mathbf{S}) = \left\{ Q = \sum_{i=1}^k \mu_i Q_i, \mu_i \geq 0, \sum \mu_i = 1 \right\}$$

is a polyhedron. Its boundary $\partial\Gamma$ consists of vertices $\Gamma_j^{(0)}$, edges $\Gamma_j^{(1)}$ and faces $\Gamma_j^{(d)}$ of different dimensions $d : 1 < d \leq n - 1$.

If the real n -vector $P = (p_1, \dots, p_n)$ is given, then the maximum and minimum of the scalar product $\langle P, Q \rangle = p_1 q_1 + \dots + p_n q_n$ on \mathbf{S} are reached at points Q_i that lie on the boundary $\partial\Gamma$.

1. Polyhedron and normal cone (2)

For each boundary element $\Gamma_j^{(d)}$ (including vertices $\Gamma_j^{(0)}$ and edges $\Gamma_j^{(1)}$), we identify the set of vectors P whose maximum $\langle P, Q \rangle$ is reached on points $Q_i \in \Gamma_j^{(d)}$. This will be its **normal cone**

$$\mathbf{U}_j^{(d)} = \left\{ P : \langle P, Q' \rangle = \langle P, Q'' \rangle > \langle P, Q''' \rangle \text{ for } Q', Q'' \in \Gamma_j^{(d)}, Q''' \in \Gamma \setminus \Gamma_j^{(d)} \right\}.$$

The vector P lies in the space \mathbb{R}_*^n , dual to the space \mathbb{R}^n .

Example 1.1 (1)

Let the set \mathbf{S} consist of three points $Q_1 = (3,0)$, $Q_2 = (0,3)$, $Q_3 = (1,1)$. The convex hull Γ is a triangle with these vertices $\Gamma_j^{(0)} = Q_j$ and three edges $\Gamma_j^{(1)}$, $j = 1,2,3$ on the plane \mathbb{R}^2 (Fig. 1.1).

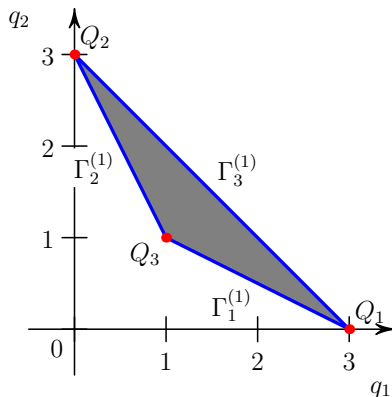


Figure 1.1: The support \mathbf{S} and the polygon Γ of Example 1.1.

Example 1.1 (2)

The edge $\Gamma_1^{(1)} = [Q_1, Q_3]$, its guide vector $R_1 = Q_3 - Q_1 = (2, -1)$. The normal vector $N_1 = -(1, 2)$ is directed from the edge $\Gamma_1^{(1)}$ outside the triangle Γ . The ray $\lambda N_1 = (-\lambda, -2\lambda)$ with $\lambda > 0$ is a normal cone $\mathbf{U}_1^{(1)}$ stretched on it.

Similarly, we find $\mathbf{U}_2^{(1)} = \lambda(-2, -1)$ and $\mathbf{U}_3^{(1)} = \lambda(1, 1)$ with $\lambda > 0$. The normal cone of the vertex $\Gamma_1^{(0)} = Q_1$ is

$$\mathbf{U}_1^{(0)} = \{P = \lambda(-1, -2) + \mu(1, 1), \lambda, \mu > 0\}.$$

Example 1.1 (3)

Similarly, we find the normal cones

$$\mathbf{U}_2^{(0)} = \{P = \lambda(-2, -1) + \mu(1, 1), \lambda, \mu > 0\}$$

and

$$\mathbf{U}_3^{(0)} = \{P = \lambda(-1, -2) + \mu(-2, -1), \lambda, \mu > 0\}.$$

They are shown in Figure 1.1.

Example 1.1 (4)

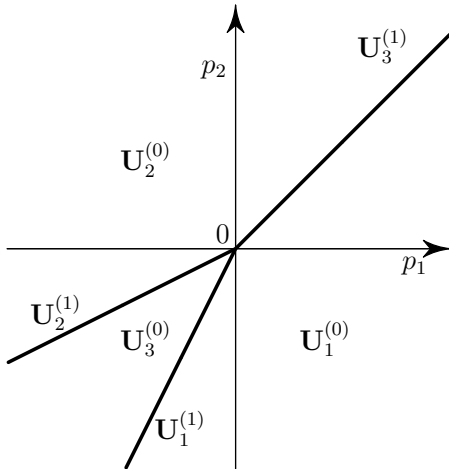


Figure 1.2: Normal cones $U_j^{(d)}$ of faces $\Gamma_j^{(d)}$ of the polygon Fig. 1.1.

Example 1.1 (5)

So, on the conjugate plane \mathbb{R}_*^2 , the rays normal to the edges of $\Gamma_j^{(1)}$ and directed outward of triangle Γ form the normal cones $\mathbf{U}_j^{(1)}$ of edges $\Gamma_j^{(1)}$ (Fig. 1.2). The sectors between them are the normal cones $\mathbf{U}_j^{(0)}$ of the vertices $\Gamma_j^{(0)}$. ■

1. Polyhedron and normal cone (continued)

Let us be interested not in the whole boundary $\partial\Gamma$, but only a part of it corresponding to some set \mathcal{K} of directions P . Then let us call the set \mathcal{K} the *cone of the problem*. It is not necessarily convex. By $\partial\Gamma(\mathcal{K})$ we denote the part of the boundary $\partial\Gamma$ for whose elements $\Gamma_j^{(d)}$ their normal cones $\mathbf{U}_j^{(d)}$ intersect with the cone of problem \mathcal{K} . Let us call $\partial\Gamma(\mathcal{K})$ — *boundary of the problem*.

Example 1.2 (1)

Let

- \mathcal{K}_1 be the upper half-plane
 $\mathcal{K}_1 = \{P = (p_1, p_2) : p_2 > 0\},$
- \mathcal{K}_2 be the right half-plane
 $\mathcal{K}_2 = \{P = (p_1, p_2) : p_1 > 0\},$
- \mathcal{K}_3 is the third quadrant
 $\mathcal{K}_3 = \{P = (p_1, p_2) : p_1 < 0\}.$

Then

- $\partial\Gamma(\mathcal{K}_1)$ is the top of the border $\partial\Gamma,$
- $\partial\Gamma(\mathcal{K}_2)$ is the right side of the boundary of $\partial\Gamma,$
- $\partial\Gamma(\mathcal{K}_3)$ is the intersection of the left and bottom side of $\partial\Gamma.$

Example 1.2 (2)

For the polygon Γ of Example 1.1, the problem boundaries are as follows:

$$\begin{aligned}\partial\Gamma(\mathcal{K}_1) &= \partial\Gamma(\mathcal{K}_2) = \Gamma_1^{(0)} \cup \Gamma_2^{(0)} \cup \Gamma_3^{(1)}, \\ \partial\Gamma(\mathcal{K}_3) &= \Gamma_1^{(0)} \cup \Gamma_2^{(0)} \cup \Gamma_3^{(0)} \cup \Gamma_1^{(1)} \cup \Gamma_2^{(1)}.\end{aligned}$$

The cones of these problems and the boundaries of the problems are shown in Fig. 1.3.

Example 1.2 (3)

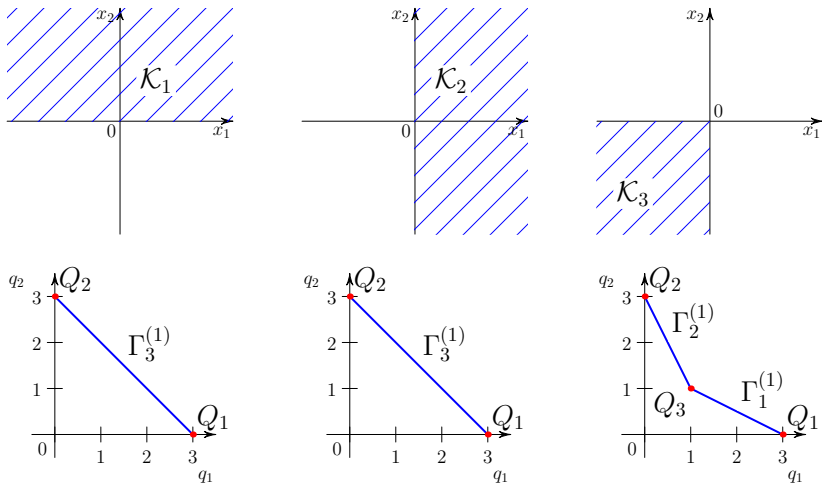


Figure 1.3: Cones \mathcal{K}_l and boundaries $\partial\Gamma(\mathcal{K}_l)$ of Problems 1.1 and 1.2.

Software for convex hull and normal cones computation

Various software is available for working with convex sets. Here we briefly describe only those programs that can be used both to compute convex hulls and to compute their normal cones.

Let's consider the following software

- *Qhull* [Barber (et al.), 1996];
- *The PolyhedralSets* for CAS Maple [Maplebook];
- packages for Sage [The Sage Developers, 2020].

Qhull

The Qhull package is used in many application software packages, both commercial and free, the Qhull package has a software interface with the Matlab scientific calculation system, GNU Octave, computer algebra systems Mathematica and Maple, libraries SciPy and geometry for programming languages Python and R respectively.

The main feature of the package is that the calculations are performed using real numbers rather than in the field of rational numbers, which is convenient when working with the Hadamard polyhedron. When calculating the Newton polyhedron, additional steps are required to bring the results of the calculations to rational values.

The PolyhedralSets (1)

Since the 2015 version, the Maple computer algebra system includes the PolyhedralSets package. It allows, in particular, to compute the convex hull of a set, to give its H - or V -representations, i.e., either as equations of hyperplanes of the boundary, or as a set of extreme points and rays, the linear combination of which gives an unbounded convex hull.

In this package all calculations are performed in the field of rational numbers, which somewhat simplifies its use for the study of the Newton polyhedron, but makes it useless when working with the Hadamard polyhedron. Note that PolyhedralSets has extremely low performance compared to Qhull.

The PolyhedralSets (2)

```
> # Вычисление нормальных конусов  
with(PolyhedralSets):  
> SFolium := [[3, 0], [0, 3], [1, 1]]:  
CH_SFolium:=ConvexHull(PolyhedralSet(SFolium)):  
Relations(CH_SFolium);  

$$\left[ -x_1 - 2x_2 \leq -3, -x_1 - \frac{x_2}{2} \leq -\frac{3}{2}, x_2 + x_1 \leq 3 \right]$$
  
> Plot(CH_SFolium);
```

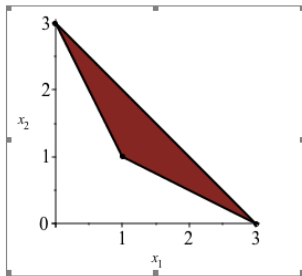


Figure 1.4: Example of computation of normal cones in the CAS Maple.

Other packages

A large selection of libraries for research in computational geometry is provided by the freeware computer algebra system Sage [The Sage Developers, 2020]. It allows you to work with libraries: cdd [Fukuda, 2002], PPL (Parma Polyhedral Library) [Bagnara (et al.), 2008], polymake, Qhull and others.

1. Polyhedron and normal cone

1.1 Software for convex hull and normal cones computation

2. Hadamard broken line method

2.1 Hadamard broken line theory

2.2 Some software for root computation

3. Plane algebraic curve

3.1 Local analysis of a simple point

3.2 Local analysis of the singular point $X^0 = 0$ and points at infinity

3.3 Real curve

3.4 Sketch of a real curve

3.5 Software for plane curve investigation

3.6 Big example

2. Hadamard broken line method (1)

Let us describe a new method for computing approximate values of roots of the polynomial

$$f_m(x) = \sum_{k=0}^m a_k x^k. \quad (2.1)$$

To do this, the points in the real plane q_1, q_2 are plotted $\check{Q}_k = (q_1, q_2) = (k, \ln |a_k|)$, $k = 0, \dots, m$, forming the **supersupport** $\check{\mathbf{S}} = \{\check{Q}_0, \dots, \check{Q}_m\}$, and their convex hull is constructed

$$\Gamma(\check{\mathbf{S}}) = \left\{ \check{Q} = \sum_{k=0}^m \mu_k \check{Q}_k, \mu_k \geq 0, \sum_{k=0}^m \mu_k = 1 \right\} \stackrel{\text{def}}{=} \mathbf{H}(f_m),$$

which is called **Hadamard's polygon** [Hadamard, 1893].

2. Hadamard broken line method (2)

The boundary $\partial\mathbf{H}$ is a broken line. Each edge $\Gamma_j^{(1)}$ and vertex $\Gamma_j^{(0)}$ of this boundary $\partial\mathbf{H}$ corresponds to a boundary subset $\mathbf{S}_j^{(d)}$ of points \check{Q}_k lying on $\Gamma_j^{(d)}$, and the truncated polynomial

$$\check{f}_j^{(d)}(x) = \sum a_k x^k \text{ by } \check{Q}_k \in \mathbf{S}_j^{(d)}. \quad (2.2)$$

The truncated polynomial contains only those term $a_k x^k$, which absolute values $|a_k x^k|$ are maximal for fixing $|x| : \ln |x| = p_1$, for then

$$\left| a_k x^k \right| = \exp \langle \check{Q}_k, (p_1, 1) \rangle,$$

and the exponent is a monotonic function.

2. Hadamard broken line method (3)

If $\Gamma_j^{(d)}$ is a vertex ($d = 0$), then the truncated polynomial (2.2) is a monomial that has no nonzero root. If $\Gamma_j^{(d)}$ is an edge ($d = 1$), then the truncated polynomial (2.2) has nonzero roots, which give approximate values for the roots of the full polynomial (2.1). Except exceptional cases, the truncated polynomials (2.2) are significantly simpler than the original polynomial (2.1), and their roots are easier to compute.

Since the vector $(p_1, 1)$ lies in the upper half-plane of the dual plane \mathbb{R}_*^2 , the cone of the problem here $\mathcal{K} = \{P = (p_1, p_2) : p_2 > 0\}$, i.e. — this is the upper half-plane. It corresponds to the upper part of the boundary $\partial\mathbf{H}$. It will be called **Hadamard's broken line** and denoted by $\widetilde{\mathbf{H}}$ [Hadamard, 1893], [Bruno, 1989, ch. IV, § 2, p. 2.1].

Example 2.1 (1)

Lets consider the polynomial

$$f(x) = 9x - 10x^3 + x^5 = x(1 - x^2)(9 - x^2) \quad (2.3)$$

with roots $x_k^0 = 0, \pm 1, \pm 3$.

Its Hadamard broken line is stretched on three vertices $(1, \ln 9)$, $(3, \ln 10)$, $(5, 0)$ and is shown in Fig. 2.1. Broken line has two edges $\Gamma_1^{(1)}$ and $\Gamma_2^{(2)}$.

Example 2.1 (2)

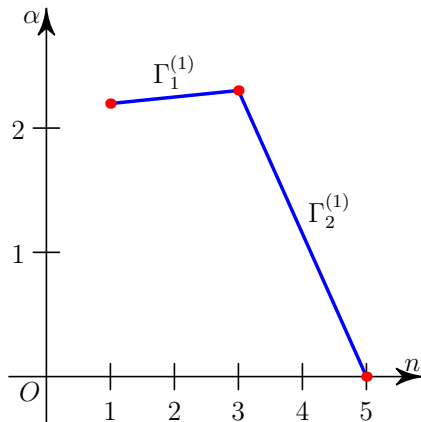


Figure 2.1: Hadamard broken line of the polynomial (2.3) of Example 2.1.

Example 2.1 (3)

Truncated polynomials

$$\check{f}_1^{(1)}(x) = 9x - 10x^3 \text{ и } \check{f}_2^{(1)}(x) = -10x^3 + x^5, \quad (2.4)$$

correspond to the edges $\Gamma_1^{(1)}$ и $\Gamma_2^{(2)}$, which roots $x_1 = 0$, $x_{2,3} = \pm\sqrt{0.9} \approx \pm 0.948683$ and $x_{4,5} = \pm\sqrt{10} \approx \pm 3.162278$ give approximate values of nonzero roots ± 1 and ± 3 of the initial polynomial $f(x)$.

To refine them, you can use Newton's method, i.e. look for the expansion of the root $x^0 = x + \varepsilon_1 + \varepsilon_2 + \dots$. For the first correction ε_1 from the equation $f(x) + \varepsilon_1 f'(x) = 0$ we obtain the value $\varepsilon_1 = -f(x)/f'(x)$. The values of the corrections ε_1 to x_k , $k = 2, \dots, 5$, and the refined values $x_k + \varepsilon_1$ are shown in Table 1. Its bottom row shows the exact roots of x_k^0 of the polynomial (2.3).

Example 2.1 (4)

Table 1: Corrections to roots of polynomials (2.4)

k	2	3	4	5
x_k	0.948683	-0.948683	3.162278	-3.162278
ε_1	0.055085	-0.055085	-0.136175	0.136175
$x_k + \varepsilon_1$	1.003768	-1.003768	3.026103	-3.026103
x_k^0	1	-1	3	-3

Example 2.2 (1)

Using the Hadamard broken line, let's find the approximate values of the roots of the polynomial

$$f(x) = 3 + 3x^2 + x^4 \quad (2.5)$$

with roots $x_k^0 = \pm \frac{1}{2} \sqrt{-6 \pm 2i\sqrt{3}} \approx \pm 0.340625 \pm 1.271230i$.
Its Hadamard broken line is stretched over three vertices $(0, \ln 3)$, $(2, \ln 3)$, $(4, 0)$.

Example 2.2 (2)

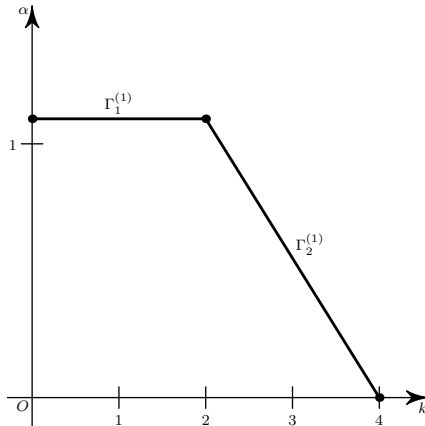


Figure 2.2: Hadamard broken line of the polynomial (2.5) of Example 2.2.

Example 2.2 (3)

It has two edges $\Gamma_1^{(1)}$ and $\Gamma_2^{(2)}$, which correspond to the truncated polynomials $\check{f}_1^{(1)}(x) = 3 + 3x^2$ and $\check{f}_2^{(1)}(x) = 3x^2 + x^4$, with corresponding roots $x_{1,2} = \pm i$ and $x_{3,4} = \pm\sqrt{3} \approx \pm 1.732051i$.

Iterations of these roots by Newton's method do not converge to exact values of the roots. For $x_{1,2}$ we obtain a periodic sequence of values with period 2 near $\pm 1.042i$. Root iterations of $x_{3,4}$ generally exhibit chaotic behavior. ■

2. Hadamard broken line method (continued)

In general, the Hadamard broken line method consists in replacing the original polynomial by a set of simpler polynomials whose roots are computed easily and give such approximations of the roots of the original polynomial that are good enough for numerical refinement by Newton's method.

Hadarnard broken line theory (1)

Let us find the reason for the distance between the exact roots of x_i^0 of the polynomial $f_m(x) = \sum_{k=0}^m a_k x^k$ and the approximate roots of x_i obtained by the Hadarnard broken line method. Assume

$$\alpha_k = \ln |a_k| \text{ and } \beta_k = \begin{cases} a_k/|a_k|, & \text{if } a_k \neq 0, \\ 0, & \text{if } a_k = 0, \end{cases}$$

where $k = 0, \dots, m$.

Then the polynomial $f_m(x)$ can be written as $f(x) = \sum_{k=0}^m \beta_k x^k e^{\alpha_k}$, where all $|\beta_k| = \pm 1$ and 0. Let's match it with the sum of $g(x, y) = \sum \beta_k x^k y^{\alpha_k}$ and consider the curve \mathcal{F} defined by Equation

$$g(x, y) = 0. \tag{2.6}$$

Hadamard broken line theory (2)

At $y \rightarrow \infty$ the branches of the curve \mathcal{F} are approximated by solutions of the truncated equations $\hat{g}_l^{(1)}(x, y) = 0$, corresponding to the edges of $\tilde{\Gamma}_l^{(1)}$ of the Hadamard broken line $\tilde{\mathbf{H}}$ of the polynomial $f_m(x)$.

If in the interval

$$y \in [e, \infty] \quad (2.7)$$

these branches do not intersect or merge, then the approximate roots of x_i obtained with the Hadamard broken line are close to the exact roots of x_i^0 of the polynomial (2.1).

Hadamard broken line theory (3)

But if in the interval $[e, \infty]$ these branches intersect or merge, then at $y = e$ their mutual placement differs from the limiting location at $y \rightarrow \infty$. The branches of the curve (2.6) can intersect or merge only at points (x^0, y^0) : $g(x^0, y^0) = \frac{\partial g}{\partial x}(x^0, y^0) = 0$, i.e., where the discriminant Δ of the polynomial g from x is zero.

Therefore, if the curve (2.6) has no points

$$(x^0, y^0) \text{ with } \Delta(x^0, y^0) = 0 \quad (2.8)$$

in the interval $[e, \infty]$, then the approximate roots x_i are close to the exact roots x_i^0 of the polynomial (2.1).

Hadamard broken line theory (4)

But if the curve (2.6) has (2.7) points (2.8) in the interval, then the approximate roots x_i may be far from the exact roots x_i^0 of the polynomial (2.1). For the proximity of real roots of x_i and x_i^0 , the absence of real non-isolated points (2.8) in the interval $[e, \infty]$ is sufficient.

Example 2.3 (continuation of example 2.1)

Here, the sum $g(x, y) = \sum \beta_k x^k y^{\alpha_k}$ for the polynomial (2.3) is

$$g(x, y) = xy^{\alpha_1} + x^3 y^{\alpha_3} - x^5,$$

where $\alpha_1 = \ln 9 \approx 2.197$, $\alpha_3 = \ln 8 \approx 2.0794$.

Assume $z = x^2$ and $h = g/x$, then we get the sum

$$h(z, y) = y^{\alpha_1} + zy^{\alpha_3} - z^2.$$

Its discriminant is $\Delta = y^{2\alpha_3} + 4y^{\alpha_1}$. It is equal to zero only at $y = 0$, obviously less than e .

So, there are no zero points of the discriminant in the interval (2.7). Therefore, the branches of the curve (2.6) at $y \rightarrow +\infty$ are also located as their intersections with the horizontal $y = e$. ■

Example 2.4 (1)

Let

$$f(x) = e + ex + x^2.$$

Then

$$g(x, y) = y + xy + x^2. \quad (2.9)$$

The discriminant of this polynomial on x is $\Delta = y^2 - 4y$. It is zero at $y = 0$ and $y = 4$. Since $y = 4$ lies in the interval (2.7), the real approximate roots $x_1 = -e$ and $x_2 = -1$ do not converge to the exact complex roots when refined by Newton's method

$$x_{1,2}^0 = -\frac{e}{2} \mp \frac{1}{2}\sqrt{e^2 - 4e} \approx -1.3591 \mp 0.9333i$$

of the polynomial $f(x)$.

Example 2.4 (2)

The iterations of the approximate roots of x_1 and x_2 are shown in Table 2, where the number k of the iteration by Newton's algorithm is given in the top row. The branch graph of the curve $g = 0$ for $y > 0$ is shown in Fig. 2.3.

Table 2:

k	0	1	2	4	5	6	7
x_1	-1.000	-2.392	-1.454	3.179	0.8141	-0.4729	-1.407
x_2	-2.718	-1.718	-0.3261	-1.264	-5.898	-3.532	-2.245

Example 2.4 (3)

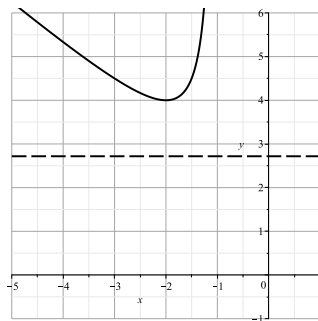


Figure 2.3: Graph of the curve branch $g = 0$ for (2.9) and $y > 0$.

The figure shows that the two approximate roots corresponding to each branch of the curve merge when $y \rightarrow 4$ and no longer reach the value of $y = e$.

Some software for root computation

There are numerous programs that allow to numerically find the roots of any polynomial with any accuracy. Any computer algebra system allows to do this numerically. For example,

- In `Mathematica`, the function `NRoots` does this;

Some software for root computation

There are numerous programs that allow to numerically find the roots of any polynomial with any accuracy. Any computer algebra system allows to do this numerically. For example,

- In `Mathematica`, the function `NRoots` does this;
- In `Python` there is a high-precision arithmetic package `mpmath`, where the function `polyroots` does the root calculation;

Some software for root computation

There are numerous programs that allow to numerically find the roots of any polynomial with any accuracy. Any computer algebra system allows to do this numerically. For example,

- In `Mathematica`, the function `NRoots` does this;
- In `Python` there is a high-precision arithmetic package `mpmath`, where the function `polyroots` does the root calculation;
- `Maple` has a special package `RootFinding` which contains functions for isolating real and complex roots with arbitrary precision.

1. Polyhedron and normal cone

1.1 Software for convex hull and normal cones computation

2. Hadamard broken line method

2.1 Hadamard broken line theory

2.2 Some software for root computation

3. Plane algebraic curve

3.1 Local analysis of a simple point

3.2 Local analysis of the singular point $X^0 = 0$ and points at infinity

3.3 Real curve

3.4 Sketch of a real curve

3.5 Software for plane curve investigation

3.6 Big example

3. Plane algebraic curve

Let $f(x_1, x_2)$ be a polynomial with real or complex coefficients. The set of solutions x_1, x_2 of the equation

$$f(x_1, x_2) = 0 \tag{3.1}$$

in $X = (x_1, x_2) \in \mathbb{R}^2$ or \mathbb{C}^2 is called a *plane algebraic curve* \mathcal{F} .

3. Plane algebraic curve

Let $f(x_1, x_2)$ be a polynomial with real or complex coefficients. The set of solutions x_1, x_2 of the equation

$$f(x_1, x_2) = 0$$

in $X = (x_1, x_2) \in \mathbb{R}^2$ or \mathbb{C}^2 is called a *plane algebraic curve* \mathcal{F} .

A point $X = X^0$, $f(X^0) = 0$ is called a *simple* point of a curve \mathcal{F} if the vector $(\partial f / \partial x_1, \partial f / \partial x_2)$ in it is nonzero. Otherwise, the point X^0 is called *singular*. By a shift, move the point X^0 to the origin of coordinates.

3.1. Local analysis of a simple point

Theorem 3.1 (Cauchy).

If at $X^0 = 0$ the derivative $\partial f / \partial x_i \neq 0$, then all solutions to the equation (3.1) near point $X^0 = 0$ are contained in the expansion

$$x_i = \sum_{k=1}^{\infty} b_k x_j^k, \quad (3.1)$$

where b_k — constants, and $j = 2 - i$.

3.2. Local analysis of the singular point (1)

Let's write the polynomial $f(X)$ as

$$f(X) = \sum f_Q X^Q \text{ by } Q \geq 0, \quad Q \in \mathbb{Z}^2, \quad (3.2)$$

where $X = (x_1, x_2)$, $Q = (q_1, q_2)$, $X^Q = x_1^{q_1} x_2^{q_2}$, $f_Q \in \mathbb{C}$ are constants. Let $\mathbf{S}(f) = \{Q : f_Q \neq 0\} \subset \mathbb{R}^2$.

3.2. Local analysis of the singular point (2)

The set \mathbf{S} is called the **support** of the polynomial $f(X)$. Let it consist of points Q_1, \dots, Q_k . The convex hull of the support $\mathbf{S}(f)$ is the set

$$\Gamma(\mathbf{S}) = \left\{ Q = \sum_{j=1}^k \mu_j Q_j, \mu_j \geq 0, \sum_{j=1}^k \mu_j = 1 \right\} \stackrel{\text{def}}{=} \mathbf{N}(f),$$

which is called **Newton's polygon**. The boundary $\partial \mathbf{N}(f)$ consists of vertices $\Gamma_j^{(0)}$ and edges $\Gamma_j^{(1)}$, where j is the number.

3.2. Local analysis of the singular point (3)

Each generalized face $\Gamma_j^{(d)}$ corresponds to its *boundary subset* $\mathbf{S}_j^{(d)} = \mathbf{S} \cap \Gamma_j^{(d)}$, its a *truncated* polynomial $\hat{f}_j^{(d)}(X) = \sum f_Q X^Q$ by $Q \in \mathbf{S}_j^{(d)}$ and its *normal cone*

$$\mathbf{U}_j^{(d)} = \{P : \langle P, Q' \rangle = \langle P, Q'' \rangle > \langle P, Q''' \rangle, Q'' \in \Gamma_j^{(d)}, Q''' \in \Gamma \setminus \Gamma_j^{(d)}\},$$

where $P = (p_1, p_2) \in \mathbb{R}_*^2$, and the plane \mathbb{R}_*^2 is conjugate to the plane \mathbb{R}^2 .

3.2. Local analysis of the singular point (4)

Let $x_1 \in \mathbb{C}$, $x_1 \rightarrow 0$ or ∞ , and $o(1)$ be a function of x_1 , which tends to zero at that. On the curve

$$x_2 = bx_1^p(1 + o(1)),$$

where $b = \text{const} \in \mathbb{C}$, $p \in \mathbb{R}$, monomial

$$f_Q X^Q = f_Q b^{q_2} \{ \exp [\langle Q, P \rangle \omega | \ln x_1 |] \} (1 + o(1)). \quad (3.3)$$

3.2. Local analysis of the singular point (5)

In this case $P = (1, p)$, and $\omega \stackrel{\text{def}}{=} \text{sgn } \ln |x_1| = \begin{cases} -1, & \text{if } x_1 \rightarrow 0, \\ 1, & \text{if } x_1 \rightarrow \infty \end{cases}$.

This means that for given P and ω the largest moduli have those monomials (3.3) of the sum (3.2) on which the value

$$\omega \langle Q, P \rangle \text{ by } Q \in \mathbf{S} \tag{3.4}$$

reaches a maximum.

3.2. Local analysis of the singular point (6)

- If $x_1 \rightarrow 0$, $\omega = -1$ and the vector $\omega P = (-1, -p)$. Hence, here the cone of the problem \mathcal{K} is the left half-plane of the plane \mathbb{R}_*^2 and the points Q with maximal values of the product (3.4) lie on the left side of the boundary $\partial\mathbf{N}$.
- If $x_1 \rightarrow \infty$, $\omega = 1$ and vector $\omega P = (1, p)$. Hence, here the cone of the problem \mathcal{K} is the right half-plane of the plane \mathbb{R}_*^2 and the points Q with the largest values of the product (3.4) lie on the right side of the boundary $\partial\mathbf{N}$.

3.2. Local analysis of the singular point (7)

We will look for solutions to the equation

$$f(X) \stackrel{\text{def}}{=} \sum_{Q \in \mathbf{S}} f_Q X^Q = 0 \quad (3.5)$$

In the form of expansion

$$x_2 = b_1 x_1^{p_1} + b_2 x_1^{p_2} + b_3 x_1^{p_3} + \cdots, \quad (3.6)$$

where $f_Q, b_k = \text{const} \in \mathbb{C}$, $Q \in \mathbb{R}^2$, $p_k = \text{const} \in \mathbb{R}$, $\omega p_k > \omega p_{k+1}$.

In these expansions, the exponents of degree p_k increase with k if $x_1 \rightarrow 0$, and decrease if $x_1 \rightarrow \infty$.

3.2. Local analysis of the singular point (8)

Theorem 3.2.

For solutions $x_2 = b_1x_1^{p_1} + b_2x_1^{p_2} + b_3x_1^{p_3} + \dots$ of equation $\sum_{Q \in \mathbf{S}} f_Q X^Q = 0$, the truncated solution

$$x_2 = b_1x_1^{p_1}$$

is the solution to the truncated equation

$$\hat{f}_j^{(d)}(X) = 0,$$

corresponding to the boundary element $\Gamma_j^{(d)}$ with the external normal vector $\omega(1, p_1) \in \mathbf{U}_j^{(d)}$.

Example 3.1 (1)

Let

$$f = x_1^3 + x_2^3 - 3x_1x_2.$$

The curve $f = 0$ is the folium of Descartes.

The polynomial support f consists of three points $Q_1 = (3, 0)$, $Q_2 = (0, 3)$, $Q_3 = (1, 1)$ (Example 1.1). Their convex hull is a triangle $\Gamma = \mathbf{N}$ with these vertices (Fig. 1.1). It has three edges $\Gamma_1^{(1)}$, $\Gamma_2^{(1)}$ and $\Gamma_3^{(1)}$ with external normals $-(2,1)$, $-(1,2)$, $(1,1)$ respectively.

The edge $\Gamma_1^{(1)}$ corresponds to the truncated equation $\hat{f}_1^{(1)} = x_1^3 - 3x_1x_2 = 0$. It has the solution

$$x_2 = \frac{1}{3}x_1^2. \quad (3.7)$$

Example 3.1 (2)

The $\Gamma_2^{(1)}$ edge corresponds to the truncated equation $\hat{f}_2^{(1)} = x_2^3 - 3x_1x_2 = 0$. It has a solution

$$x_2 = \pm\sqrt{3x_1}. \quad (3.8)$$

The branches (3.7) and (3.8) belong to the neighborhood of zero $x_1 = x_2 = 0$ because the outer normals of the corresponding edges $\Gamma_1^{(1)}$ and $\Gamma_2^{(1)}$ have both coordinates negative.

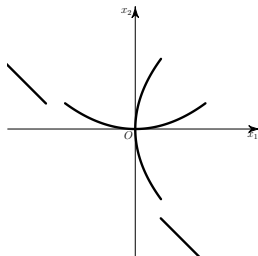
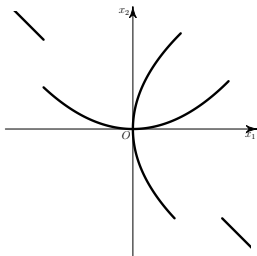
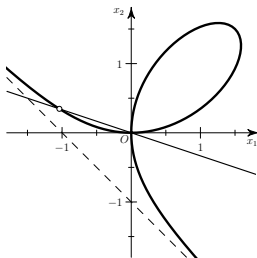
Example 3.1 (3)

Finally, the edge $\Gamma_3^{(1)}$ corresponds to the truncated equation $\hat{f}_3^{(1)} = x_1^3 + x_2^3 = (x_1 + x_2)(x_1^2 - x_1x_2 + x_2^2) = 0$. It has one real solution

$$x_2 = -x_1.$$

It belongs to the neighborhood of infinity $x_1 = x_2 = \infty$, because the external normal to the edge $\Gamma_3^{(1)}$ has both coordinates positive.

Example 3.1 (4)



3.2. Local analysis of singular point (cont.) (1)

The truncated equation $\hat{f}_j^{(1)}(X) = 0$ uniquely determines the sign of ω and the index of degree p_1 . If in the sum (3.2) all vector indexes of degree $Q = (q_1, q_2)$ have rational components q_1 and q_2 , then the index p_1 is rational. For the coefficient b_1 we obtain the algebraic equation

$$\hat{f}_j^{(1)}(1, b_1) = 0.$$

Theorem 3.3.

For the polynomial equation $\sum_{Q \in \mathbf{S}} f_Q X^Q = 0$, all solutions $x_2(x_1)$ are expanded into a series of the form

$$x_2 = b_1 x_1^{p_1} + b_2 x_1^{p_2} + b_3 x_1^{p_3} + \cdots,$$

where all exponents of degree p_k are rational numbers with a common denominator.

3.2. Local analysis of singular point (cont.) (2)

For the neighborhood of the point $X = 0$, the theorem 3.3 is that of V. Puiseux, 1850, i.e., for the cone problem $\mathcal{K} = \{P = (p_1, p_2) : p_1, p_2 < 0\}$. The corresponding part (lower left) of the boundary $\partial\mathbf{N}$ is called *Newton's broken line*. The expansions of the theorem 3.1 converge, so all expansions (3.6) for solutions of polynomial equations (3.5) also converge.

Real curve (1)

If the polynomial $f(X) = \sum f_Q X^Q$ has all coefficients of f_Q real, then the equation $f(X) = 0$ defines a real curve \mathcal{F} on the plane \mathbb{R}^2 . For it in the expansion $x_2 = \sum_{k=1}^{\infty} b_k x_1^k$ of Theorem 3.1 all the coefficients b_k are real, and from the expansion $x_2 = b_1 x_1^{p_1} + b_2 x_1^{p_2} + \dots$ of Theorem 3.3 only those in which all coefficients b_k are real have sense. Since they are computed sequentially, at each step only real coefficients of b_k should be left, and the complex coefficients should be discarded.

Real curve (2)

At the singular point $X = 0$ the expansion (3.1) starts with the quadratic terms

$$f = g_{20}x_1^2 + g_{11}x_1x_2 + g_{02}x_2^2 + \dots \stackrel{\text{def}}{=} g_2(x_1, x_2) + \dots$$

Theorem 3.4.

If the discriminant $\Delta = g_{11}^2 - 4g_{20}g_{02}$ of the quadratic form g_2 is less than zero, then the curve F has no real branches passing through the critical point $X = 0$.

Real curve (3)

Let there be two polynomials of degree 2 and 3, respectively:

$$g = g_0x^2 + g_1x + g_2, \quad h = h_0x^3 + h_1x^2 + h_2x + h_3.$$

If polynomials g and h have a common root, then the resultant $\text{Res}(g, h)$ is identically zero.

Real curve (4)

Theorem 3.5.

Let at the singular point $X = 0$

$$f = f_2(X) + f_3(X) + \dots,$$

where $f_k(X)$ — are homogeneous forms of degree k by x_1, x_2 , the form $f_2 \not\equiv 0$ and its discriminant $\Delta = 0$, and the resultant $\text{Res}(f_2, f_3) \neq 0$, then near point $X = 0$ the curve F — is a semicubic parabola (cusp) of the form

$$x_2 = \alpha x_1 \pm \beta x_1^{3/2} + \dots,$$

where α is a multiple root of the polynomial $f_2(1, \alpha)$.

Sketch of a real curve (1)

Let the polynomial $f(X)$ have all coefficients real, then on the real plane $X \in \mathbb{R}^2$ we can draw all its branches using the local analysis described above. Let us divide this procedure into several steps.

Step 1.

We expand the polynomial $f(X)$ into polynomial multipliers using various computer algebra algorithms. Then we construct curve sketches for each irreducible factor $f_l(X)$ separately.

Sketch of a real curve (2)

Step 2.

Find all real finite singular points of X^0 of the curve $f = 0$ in which

$$f(X^0) = 0, \quad \frac{\partial f}{\partial x_1}(X^0) = 0, \quad \frac{\partial f}{\partial x_2}(X^0) = 0,$$

using the exclusion method or the Gröbner basis method.

Step 3.

Near each singular point X^0 , find all real branches of the form (3.6) by moving it to the origin and using methods of subsections 2 and 3

.

Sketch of a real curve (3)

Step 4.

Find the intersection points of the curve with the axes $x_1 = 0$ and $x_2 = 0$ as solutions of the equations $f(0, x_2) = 0$ and $f(x_1, 0) = 0$, using the methods of section 2 and refining them by the Theorem 3.1.

Step 5.

Find the finite intersection points of the curve with infinities $x_1 = \infty$ and $x_2 = \infty$ by truncated equations with $P = (1, 0)$ and $P = (0, 1)$. For each of these, compute the initial terms of the (3.6) type expansion.

Sketch of a real curve (4)

Step 6.

Find the branches of the curve at $x_1 \rightarrow 0$, $x_2 \rightarrow \infty$, using part of the $\partial\mathbf{N}$ boundary with the cone problem $\mathcal{K}_1 = \{P : p_1 < 0, p_2 > 0\}$. Similarly, when $x_1 \rightarrow \infty$, $x_2 \rightarrow 0$ — with the problem cone $\mathcal{K}_2 = \{P : p_1 > 0, p_2 < 0\}$.

Step 7.

Find the branches of the curve at $x_1, x_2 \rightarrow \infty$ using part of the boundary $\partial\mathbf{N}$ with the cone problem $\mathcal{K}_3 = \{P : p_1 > 0, p_2 > 0\}$.

Step 8.

Connect the found pieces of curve branches, given that outside the special points of X^0 the curve branches do not intersect.

Software for plane curve investigation (1)

The CAS Maple system has an excellent package `algcurves` that allows to study planar algebraic curves: build their sketches with high precision, calculate their genus, find singular points, for curves of genus 0 find rational parameterization, for elliptic curves bring to Weierstrass normal form. The package allows to construct a sketch of the curve $f(x, y) = 0$ by numerical integration of the differential equation $f'_x + f'_y y' = 0$ for some set of initial conditions defined by points in which at least one of the partial derivatives of the function $f(x, y)$ is equal to zero.

Software for plane curve investigation (2)

Since version 12, Wolfram Mathematica has included an `AsymptoticSolve` procedure that implements an asymptotic representation of solutions to equations or systems of equations (not necessarily algebraic) in the form of either Taylor, Laurent, or Puiseux series near finite or infinite points. If the point is singular, the procedure tries to calculate the asymptotic expansions of all branches. In this case we can specify that we should restrict ourselves to real expansions only.

Asymptotic solution of an algebraic equation

Alexander Bruno

Professor and Head
Department of Singular Problems
Keldysh Institute of Applied Mathematics of RAS, Moscow

Polynomial Computer Algebra, 2011

Computation of asymptotic solution of an algebraic equation

A. B. Batkhin

Department of Singular Problems
Keldysh Institute of Applied Mathematics RAS, Moscow

Polynomial Computer Algebra, 2011

$$g(Q) =$$

$$\begin{aligned} & 512z^6 - 4352z^5y - 768z^5x + 14848z^4y^2 + 5376z^4yx + 512z^4x^2 - 25408z^3y^3 - \\ & - 14656z^3y^2x - 2752z^3yx^2 - 192z^3x^3 + 21800z^2y^4 + 19168z^2y^3x \\ & + 5360z^2y^2x^2 + 736z^2yx^3 + 40z^2x^4 - 7500zy^5 - 11700zy^4x - 4376zy^3x^2 \\ & - 904zy^2x^3 - 92zyx^4 - 4zx^5 + 2500y^5x + 1200y^4x^2 \\ & + 344y^3x^3 + 48y^2x^4 + 4yx^5 - 256z^5 + \\ & + 2880z^4y + 1344z^4x - 14976z^3y^2 - 6720z^3yx - 1344z^3x^2 + \\ & + 37928z^2y^3 + 13816z^2y^2x + 5144z^2yx^2 + 456z^2x^3 - \\ & - 45120zy^4 - 14464zy^3x - 6784zy^2x^2 - 1152zyx^3 - 64zx^4 + \\ & + 20250y^5 + 6490y^4x + 3156y^3x^2 + 740y^2x^3 + 82yx^4 + 2x^5 + \\ & + 1872z^4 + 2016z^3y - 5088z^3x - 35496z^2y^2 + 15888z^2yx + \\ & + 2200z^2x^2 + 67608zy^3 - 12936zy^2x - 5176zyx^2 - 344zx^3 - \\ & - 37827y^4 + 828y^3x + 2782y^2x^2 + 412yx^3 + 13x^4 - 13824z^3 + \\ & + 62208z^2y + 6912z^2x - 93312zy^2 - 20736zyx - 1152zx^2 + 46656y^3 \\ & + 15552y^2x + 1728yx^2 + 64x^3. \end{aligned}$$

Big example (1)

Here we investigate the algebraic curve $f(y, z) \stackrel{\text{def}}{=} g(x, y, z)|_{x=0} = 0$ according to the scheme of the Subsection 4, where

$$\begin{aligned} f(y, z) = & -7500y^5z + 20250y^5 + 21800y^4z^2 - 45120y^4z - \\ & - 37827y^4 - 25408y^3z^3 + 37928y^3z^2 + 67608y^3z + \\ & + 46656y^3 + 14848y^2z^4 - 14976y^2z^3 - 35496y^2z^2 - \\ & - 93312y^2z - 4352yz^5 + 2880yz^4 + 2016yz^3 + 62208yz^2 + \\ & + 512z^6 - 256z^5 + 1872z^4 - 13824z^3 \quad (3.9) \end{aligned}$$

Big example (2)

Step 1.

The polynomial (3.9) is factorized into two multipliers: the linear $3z - 2y$ and the fifth degree polynomial

$$\begin{aligned} h(y, z) = & 2500y^4z - 6750y^4 - 5600y^3z^2 + 10540y^3z + \\ & + 12609y^3 + 4736y^2z^3 - 5616y^2z^2 - 14130y^2z - \\ & - 15552y^2 - 1792yz^4 + 1248yz^3 + 2412yz^2 + 20736yz + \\ & + 256z^5 - 128z^4 + 936z^3 - 6912z^2 \quad (3.10) \end{aligned}$$

Next we consider only the curve $\mathcal{F} : h(y, z) = 0$.

Big example (3)

Step 2.

Find the singular points of the curve by solving the system $h = \partial h / \partial y = \partial h / \partial z = 0$ and calculating the Gröbner basis \mathcal{GBJ} for it. The original system has a finite number of singular points. Their coordinates can be easily determined by calculating two Gröbner bases for orders $y > z$ and $z > y$, respectively, and writing out the first polynomials of the resulting bases:

$$y^4 (y - 14) (8000y^3 - 50832y^2 + 111321y - 84672)^2 \quad (3.11)$$

$$z^4 (z - 27) (2560z^3 - 22176z^2 + 63909z - 61344)^2 \quad (3.12)$$

The cubic polynomials in (3.11) and (3.12) each have one real root y_3, z_3 respectively, hence the special points are three: $(y_1, z_1) = (0, 0)$, $(y_2, z_2) = (14, 27)$ and (y_3, z_3) .

Big example (4)

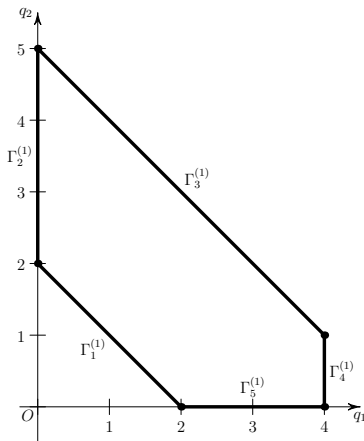
Step 3.

Find all real branches in the neighborhood of each of the special points (y_i, z_i) , $i = 1, 2, 3$.

a) The expansion at the point $(y_1, z_1) = (0, 0)$

Here the cone of the problem is $\mathcal{K}_3 = \{P = (p_1, p_2) < 0\}$.

Newton polygon of the polynomial (3.10) on the right picture.



Big example (5)

Expansion of curve \mathcal{F} at singular point $(0, 0)$

In [24]: $F = 2500 y^4 z - 6750 y^4 - 5600 y^3 z^2 + 10540 y^3 z + 12609 y^3 + 4736 y^2 z^3 - 5616 y^2 z^2 - 14130 y^2 z - 15552 y^2 - 1792 y z^4 + 1248 y z^3 + 2412 y z^2 + 20736 y z + 256 z^5 - 128 z^4 + 936 z^3 - 6912 z^2$

executed in -642ms, finished 17:33:52 2021-04-22

Out[24]: $-15552 y^2 + 12609 y^3 - 6750 y^4 + 20736 y z - 14130 y^2 z + 10540 y^3 z + 2500 y^4 z - 6912 z^2 + 2412 y z^2 - 5616 y^2 z^2 - 5600 y^3 z^2 + 936 z^3 + 1248 y z^3 + 4736 y^2 z^3 - 128 z^4 - 1792 y z^4 + 256 z^5$

In [21]: $rs1 = \text{AsymptoticSolve}[F == 0, \{z, 0\}, \{y, 0, 3\}, \text{Reals}, \text{Direction} \rightarrow \text{"FromAbove"}];$
 $rs1[[1]]$
 $rs1[[2]]$

executed in -21ms, finished 17:33:30 2021-04-22

Out[22]:
$$\left\{ z \rightarrow \frac{3y}{2} - \frac{y^2}{24} - \frac{y^{5/2}}{72} - \frac{41y^3}{3456} \right\}$$

$$\left\{ z \rightarrow \frac{3y}{2} - \frac{y^2}{24} + \frac{y^{5/2}}{72} - \frac{41y^3}{3456} \right\}$$

Big example (6)

b) The expansion in the point $(y_2, z_2) = (14, 27)$

Compute the quadratic form h_2 at the singular point (y_2, z_2) :

$$h_2(y_2 + \eta, z_2 + \zeta) = 243\eta^2 - 256\eta\zeta + 68\zeta^2.$$

Its discriminant is $\Delta = -560$, hence, by theorem 3.4 there are no real branches, and the singular point is the intersection point of complex branches where their imaginary parts are zeroed.

Big example (7)

Expansion of curve \mathcal{F} at singular point (14, 27)

```
In [17]: rs2 = AsymptoticSolve[F == 0, {z, 27}, {y, 14, 1}, Direction -> "FromBelow"];  
rs2[[1]]  
rs2[[2]]
```

executed in 130ms, finished 17:30:18 2021-04-22

Out[18]:

$$\left\{ z \rightarrow 27 + \frac{1}{34} \left(64 - i \sqrt{35} \right) (-14 + y) \right\}$$
$$\left\{ z \rightarrow 27 + \frac{1}{34} \left(64 + i \sqrt{35} \right) (-14 + y) \right\}$$

Big example (8)

c) Expansion at a point $(y_3, z_3) \approx (2.342, 3.242)$

As stated above, the coordinates of (y_3, z_3) of this point are the roots of the cubic polynomials

$$\begin{aligned} 8000 y^3 - 50832 y^2 + 111321 y - 84672, \\ 2560 z^3 - 22176 z^2 + 63909 z - 61344. \end{aligned} \tag{3.13}$$

These polynomials have one real and one pair of complex-conjugate roots each, whose values can be written in cubic radicals. However, it is very inconvenient to perform symbolic calculations in this form because of the cumbersomeness of the resulting expressions.

Big example (9)

The most general way to analytically compute asymptotic expansions of branches of a curve is to compute by modulo of the ideal \mathcal{J} defining the singular point (y_3, z_3) . All calculations are performed using the Gröbner basis of this ideal with a priori specified lexicographic order of variables. In our case, we use the Groebner package of the Maple system and the procedures `Basis` and `NormalForm` from it.

We denote the coordinates near the singular point (y_3, z_3) by η, ζ :
 $y = y_3 + \eta, z = z_3 + \zeta$.

Big example (10)

The zero-dimensional ideal \mathcal{J} defining the singular point (y_3, z_3) consists of polynomials (3.13) and derivatives h'_y, h'_z . Its Gröbner basis \mathcal{GBJ} with lexical order $y_3 < z_3$ has the form

$$\mathcal{GBJ} = \{8000y_3^3 - 50832y_3^2 + 111321y_3 - 84672, \\ -1000y_3^2 + 3579y_3 + 1416z_3 - 7488\}.$$

Big example (11)

For the singular point (y_3, z_3) , the first nondegenerate form is the quadratic form h_2 near the point (y_3, z_3) of the Taylor series:

$$\begin{aligned} h_2(y_3 + \eta, z_3 + \zeta) = & \\ & = \left(\frac{775125y_3^2}{472} - \frac{26927775y_3}{3776} + \frac{496125}{59} \right) \eta^2 + \\ & + \left(-\frac{119565y_3^2}{59} + \frac{3588165y_3}{472} - \frac{476280}{59} \right) \eta\zeta + \\ & + \left(\frac{327213y_3^2}{295} - \frac{10769787y_3}{2360} + \frac{1555848}{295} \right) \zeta^2. \quad (3.14) \end{aligned}$$

Its discriminant modulo of the ideal is zero.

Big example (12)

Calculate the cubic form h_3 near the point (y_3, z_3) :

$$\begin{aligned}h_3(y_3 + \eta, z_3 + \zeta) &= \\&= \left(\frac{605211}{59} - \frac{1196255y_3}{118} + \frac{455000y_3^2}{177} \right) \eta^3 + \\&+ \left(\frac{195000y_3^2}{59} - \frac{767820y_3}{59} + \frac{728442}{59} \right) \eta^2\zeta + \\&+ \left(-\frac{321600y_3^2}{59} + \frac{1238940y_3}{59} - \frac{1205532}{59} \right) \eta\zeta^2 + \\&\quad + \left(\frac{267872y_3^2}{177} - \frac{327988y_3}{59} + \frac{317112}{59} \right) \zeta^3\end{aligned}$$

Big example (13)

and their resultant is equal to a constant multiplier

$$\begin{aligned}\Omega(h_2, h_3) = & -940188378783151651824312 y_3^2 + \\ & + 4271494861559694739143761 y_3 - \\ & - 4838779028819532203074752 \approx 8.06 \cdot 10^{21},\end{aligned}$$

i.e., it is non-zero. Hence, theorem 3.5 is applicable.

Find the quadratic form (3.14) in the form $a(\zeta - b\eta)^2$ in order to obtain a polynomial on y_3 substitution of the variables $\zeta = \tau + b\eta$. We obtain a branch expansion in the neighborhood of the point (y_3, z_3) in the form

$$\zeta = b\eta \pm \sqrt{\frac{c}{a}} \eta^{3/2}.$$

Big example (14)

Calculations on the modulus of the ideal \mathcal{GBJ} lead to the following results:

$$\begin{aligned} a &= 216 (96952y_3^2 - 398881y_3 + 460992) \approx \\ &\approx 1.266453321 \cdot 10^7, \end{aligned}$$

$$\begin{aligned} b &= \frac{16600y_3^2}{3717} - \frac{34973y_3}{2065} + \frac{9557}{590} \approx \\ &\approx 1.03264746, \end{aligned}$$

$$\begin{aligned} c &= 8 (133000y_3^2 + 479793y_3 - 1858752) \approx \\ &\approx -38045.768 \quad . \end{aligned}$$

Big example (15)

Since $c < 0$, then $\eta < 0$ and the expansion in the variables y, z takes the form

$$z = 0.823545771 + 1.03264746 y + 0.05480984628 \times \\ \times (2.342744299 - y)^{3/2}$$

at $y \leq y_3 \approx 2.342744299$.

Step 8

Now there is enough information to construct a sketch of the curve, which is shown in Figure 3.1. It shows the graph of the curve \mathcal{F} except for the point $(y_2, z_2) = (14, 27)$. Near the singular points (y_1, z_1) and (y_3, z_3) the dotted line shows the branches calculated approximately, and the asymptote $z = 2.7$.

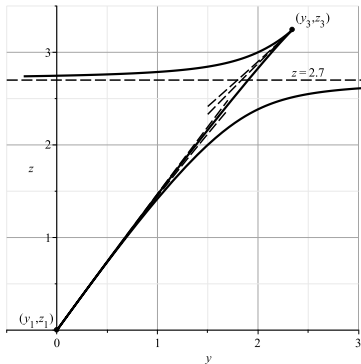









Figure 3.1:

References (1)

-  *Bagnara R., Hill P. M., Zaffanella E.* The Parma Polyhedra Library: Toward a Complete Set of Numerical Abstractions for the Analysis and Verification of Hardware and Software Systems. // Science of Computer Programming. 2008. Vol. 72, no. 1/2. P. 3–21.
-  *Barber C. B., Dobkin D. P., Huhdanpaa H. T.* The Quickhull algorithm for convex hulls. // ACM Trans. on Mathematical Software. 1996. Vol. 22, no. 4. P. 469–483. DOI: 10.1145/235815.235821.
-  *Bruno A. D.* Local Methods in Nonlinear Differential Equations. Berlin – Heidelberg – New York – London – Paris – Tokyo: Springer–Verlag, 1989.

References (2)

-  *Bruno A. D., Batkhin A. B.* Introduction to nonlinear analysis of algebraic equations. // KIAM Preprints. 2020. No. 87. DOI: 10.20948/prepr-2020-87. URL: https://keldysh.ru/papers/2020/prep2020_87.pdf ; (in Russian).
-  *Fukuda K.* cdd, cddplus and cddlib homepage. 2002. URL: http://www.cs.mcgill.ca/~fukuda/software/cdd_home/cdd.html.
-  *Hadamard J.* Etude sur les propriétés des fonctions entières et en particulier d'une fonction considérée par Riemann. // Journal de mathématiques pures et appliquées 4^e série. 1893. T. 9. P. 171-216.
-  *The Sage Developers.* SageMath, the Sage Mathematics Software System (Version 9.1.1). 2020. DOI: 10.5281/zenodo.4066866. <https://www.sagemath.org>.

Thanks for your attention!