On stratification of state space by orbit types and hierarchy of classicality indicators

Vahagn Abgaryan, Arsen Khvedelidze and Astghik Torosyan

The representation of finite-dimensional quantum systems in a phase space [1] inevitably leads to the problem of negativity of probability distributions defined over the phase space [2]. This negativity is a sign of manifestation of "quantumness" of a system and can be used to define quantitative characteristics of states [3]. Following this idea and basing on the algebraic method of construction of the Wigner functions of $N-$level quantum systems [4, 5], we introduce the global indicator of classicality $\mathcal{Q}_N$ [6, 7, 8] defined as a relative volume of a subspace $\mathfrak{P}_N^{(+)} \subset \mathfrak{P}_N$ of the state space $\mathfrak{P}_N$, where the Wigner quasiprobability distribution is positive. In the present report we analyse a refined hierarchy of measures of classicality corresponding to a natural *stratification of state space* $\mathfrak{P}_N$ *by the unitary orbit types.* The adjoint action of $SU(N)$ group on density matrices $\varrho \in \mathfrak{P}_N$,

$$g \cdot \varrho = g \varrho g^\dagger \,, \qquad g \in SU(N)\,, \tag{1}$$

induces the state space decomposition into the strata:

$$\mathfrak{P}_N = \bigcup_{\text{orbit types}} \mathfrak{P}_{[H_\alpha]}\,. \tag{2}$$

The components of decomposition (2) are determined by the *isotropy group* H_ϱ of a point $\varrho \in \mathfrak{P}_N$,

$$\mathfrak{P}_{[H_\alpha]} := \left\{ \varrho \in \ \mathfrak{P}_N \,|\ H_\varrho \text{ is conjugate to } H_\alpha \right\}. \tag{3}$$

Having in mind the above stratification, it is natural to define *global indicator of classicality* $\mathcal{Q}_N[H_\alpha]$ *of states over a given stratum* as the ratio:

$$\mathcal{Q}_N[H_\alpha] = \frac{\mathrm{Vol}\left(\mathfrak{P}^{(+)}_{[H_\alpha]}\right)}{\mathrm{Vol}\left(\mathfrak{P}_{[H_\alpha]}\right)}\,, \tag{4}$$

where $\mathfrak{P}^{(+)}_{[H_\alpha]}$ is the subset of stratum $\mathfrak{P}_{[H_\alpha]}$ where the Wigner quasiprobability distribution is non-negative. In the definition (4) the volumes are evaluated with respect to the Riemannian metric on $\mathfrak{P}_{[H_\alpha]}$ induced by the stratification embedding. In order to exemplify the introduced indicator of classicality, a detailed evaluation of the corresponding $\mathcal{Q}_3$ over all strata of qutrit state space is given. Aiming to analyse dependency of $\mathcal{Q}_3[H_\alpha]$ on the Riemannian volume measure on state space $\mathfrak{P}_N$, the Hilbert-Schmidt and two monotone [9, 10], the Bures and the Bogoliubov-Kubo-Mori metrics [11] have been used.

References

[1] M. Hillery, R.F. O'Connell, M.O. Scully and E.P. Wigner, *Distribution functions in physics: Fundamentals*, Physics Reports, **106**(3), 121-167, 1984.

[2] R.L. Hudson, *When is the Wigner quasi-probability density non-negative?*, Reports on Mathematical Physics, **6**(2), 249-252, 1974.

[3] A. Kenfack and K. Zyczkowski, *Negativity of the Wigner function as an indicator of non-classicality*, J. Opt. B: Quantum Semiclass. Opt. **6**, 396-404, 2004.

[4] V. Abgaryan and A. Khvedelidze, *On families of Wigner functions for N-level quantum systems*, *https://arxiv.org/pdf/1708.05981.pdf* (2018).

[5] V. Abgaryan, A. Khvedelidze and A. Torosyan, *On the moduli space of Wigner quasiprobability distributions for N-dimensional quantum systems*, J. Math. Sci. **240**, 617-633, 2019.

[6] V. Abgaryan, A. Khvedelidze and A. Torosyan, *The Global indicator of classicality of an arbitrary N-Level quantum system*, J. Math. Sci. **251**, 301-314, 2020.

[7] N. Abbasli, V. Abgaryan, M. Bures, A. Khvedelidze, I. Rogojin and A. Torosyan, *On measures of classicality/quantumness in quasiprobability Representations of finite-dimensional quantum systems*, Physics of Particles and Nuclei, **51**, 443-447, 2020.

[8] V. Abgaryan, A. Khvedelidze and I. Rogojin, *On overall measure of non-classicality of N-level quantum system and its universality in the large N limit*, Lecture Notes in Computer Science, **12563**, 244-255, 2021.

[9] E. Morozova and N. Chentsov, *Markov invariant geometry on state manifolds*, J. Math. Sci. **56**, 2648-2669, 1991. (Translated from: Itogi Nauki i Tekhniki, **36**, 69-102, 1990.)

[10] D. Petz, *Monotone metrics on matrix spaces*, Linear Algebra and its Applications **244**: 81-96, 1996.

[11] M.M. Deza and E. Deza, *E*ncyclopedia of Distances, Springer-Verlag, Berlin, Heidelberg (2009).

Vahagn Abgaryan
Laboratory of Information Technologies
Joint Institute for Nuclear Research
141980 Dubna, Russia
e-mail: `vahagnab@googlemail.com`

Arsen Khvedelidze
A Razmadze Mathematical Institute
Iv. Javakhishvili, Tbilisi State University
Tbilisi, Georgia
Institute of Quantum Physics and Engineering Technologies
Georgian Technical University
Tbilisi, Georgia
Laboratory of Information Technologies
Joint Institute for Nuclear Research
141980 Dubna, Russia
e-mail: akhved@jinr.ru

Astghik Torosyan
Laboratory of Information Technologies
Joint Institute for Nuclear Research
141980 Dubna, Russia
e-mail: astghik@jinr.ru