Lüneburg lens and calculation of special functions in CAS

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Abstract. The classical Lüneburg lens scattering problem has a solution expressed in terms of the Whittaker and Heun functions. The difficulties of finding these functions in CAS Maple are described. The software package «Fucsh for Sage» for working with Fuchsian singular points of linear differential equations is presented, in which the Maple results are verified.

Special functions are considered in theoretical and mathematical physics as solutions of linear ordinary differential equations, usually having singular points. An example of such classical problem we can investigate the scattering of electromagnetic waves by a Lüneburg lens [1]. This problem is reduced to solving of two ordinary differential equations:

$$\frac{d}{dr}\frac{1}{\mu}\frac{du_n}{dr} + \left[k^2\varepsilon - \frac{n(n+1)}{\mu r^2}\right] u_n = 0, \tag{1}$$

$$\frac{d}{dr}\frac{1}{\varepsilon}\frac{dv_n}{dr} + \left[k^2\mu - \frac{n(n+1)}{\varepsilon r^2}\right] v_n = 0. \tag{2}$$

Here ε, μ are dielectric and magnetic permeability of filling, depending only on the radius. In particular, for Lüneburg case we have

$$\varepsilon = 2 - r^2, \quad \mu = 1.$$

In the CAS Maple'2019 [2], particular solutions of these equations, bounded at zero, are expressed as follows:

$$u_n = \frac{1}{\sqrt{r}} \operatorname{WhittakerM}\left(\frac{k}{2}, \frac{2n+1}{4}, kr^2\right),$$

and

$$v_n = e^{kr^2/2} \operatorname{HeunC}(2k, n+1/2, -2, k^2, -k^2+3/4, r^2/2) r^{n+1}. \tag{3}$$

Here WhittakerM is Whittaker's function, HeunC is confluent Heun's function [3]. It should be noted that the class of functions ε, μ for which the equations are simultaneously solvable in Maple'2019 is very small. For Lüneburg case the values

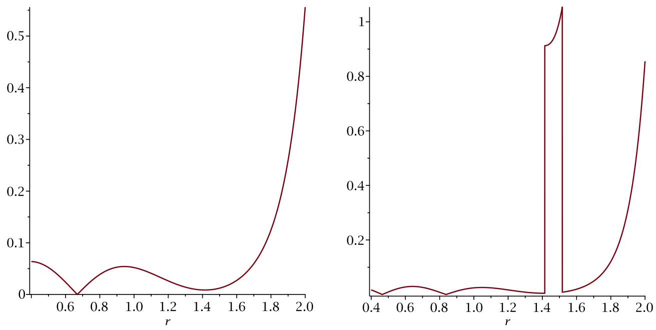

FIGURE 1. Function graphs $|v_n(r)|$ in Maple'2019, a.) $k = 5$, $n = 1$ (left); b.) $k = 7$, $n = 1$ (right).

of these functions, calculated in Maple, are sometimes very big and the graphs have numerical artifacts.

To compare the results of Maple and the results from independent source we wrote the routine «Fucsh for Sage» in CAS Sage [4]. This package allows to find solutions of linear differential equations with Fuchsian singular points in the form of Frobenius series.

The solutions to the equation (1) are displayed by the Maple quite correctly even if we have very big values for Whittaker's function. This was established by independently calculating them using the Frobenius series in CAS Sage.

The equation (2) has 4 singular points: $r = 0$, $r = \pm\sqrt{2}$, and $r = \infty$. Therefore, its solutions are expressed in terms of special functions of the Heun class [3]. The solutions of the equation (2) are bounded in the vicinity of the Fuchsian singular point $r = \sqrt{2}$ and are real for all real values of r. At the same time, for arbitrary values of the parameters k and n, the graphs of the functions $v_n(r)$ break off at the singular point $r = \sqrt{2}$. Substitution of the value $r = \sqrt{2}$ into the expression for $v_n(r)$ in Maple gives the result `Float(infinity) + Float(infinity)*I`, which is incorrect. In this case, the graph of the function $|v_n(r)|$ can be smoothly extended beyond the singular point (fig. 1, left). It should be noted that it is possible to continue beyond the singular point, accompanied by a local loss of the smoothness (fig. 1, right). It is especially important to state the presence of a range of parameters k and n, in which the graph of the function $v_n(r)$ is transmitted obviously incorrectly even with $r < \sqrt{2}$, fig. 2, left. On the other hand, our package give the correct values of the functions v_n for cases when they are displayed incorrectly (fig. 2).

Examples of erroneous calculation of special functions in Maple'2019 are presented. In CAS Sage implements an algorithm for finding these functions in the form of Frobenius series. The written software package makes it possible to find solutions to linear ordinary in some cases when their solution is not expressed in terms of known special functions.

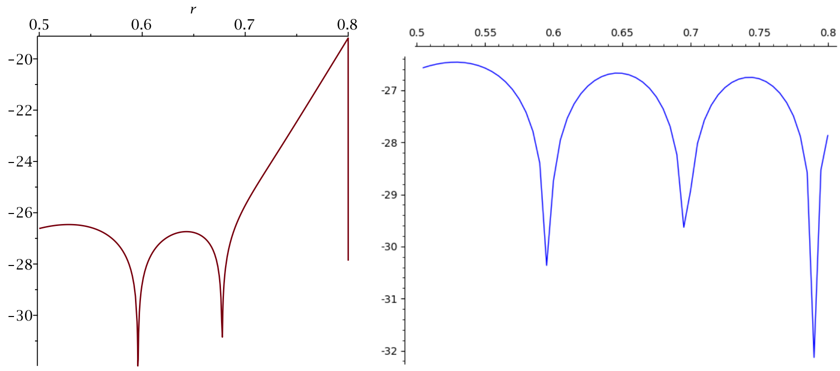

FIGURE 2. Function graphs $\ln|v_n(r)|$ for case $k = 40$, $n = 25$ in Maple (left) and in "Fuchs for Sage" (right).

References

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