Lüneburg lens and calculation of special functions in CAS (PCA’2021)

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Introduction

We consider the scattering of plane monochromatic electromagnetic wave on dielectric ball filled by optically inhomogeneous matter. For a spherically symmetric filling, the scattered field can be described using two series in spherical functions. To find the expansion coefficients, it is necessary to solve two linear ordinary differential equations.

We describe the difficulties of doing this path in CAS. As an example, we use the Lüneburg lens problem, the solution of which was obtained relatively recently [Lock, 2008].

Scattered electromagnetic field

The scattered field potentials are represented as series:

$$U = \sum_{n=1}^{\infty} u_n(r) {P_n}^{(1)}(\cos\theta)\sin\varphi$$

$$V = \sum_{n=1}^{\infty} v_n(r) {P_n}^{(1)}(\cos\theta)\cos\varphi$$

Here ${P_n}^{(1)}(\cos\theta)$ - associated Legendre functions,

n - harmonic number.

We must determine the amplitude factors $u_n(r)$, $v_n(r)$ for $n = \overline{1,100}$.

Differential equations

$$\frac{d}{dr}\frac{1}{\mu(r)}\frac{du_n}{dr}+\left[k^2\varepsilon(r)-\frac{n(n+1)}{\mu(r)r^2}\right]u_n=0$$

$$\frac{d}{dr}\frac{1}{\varepsilon(r)}\frac{dv_n}{dr}+\left[k^2\mu(r)-\frac{n(n+1)}{\varepsilon(r)r^2}\right]v_n=0$$

It is necessary to obtain analytical solutions of these two equations **simultaneously** .

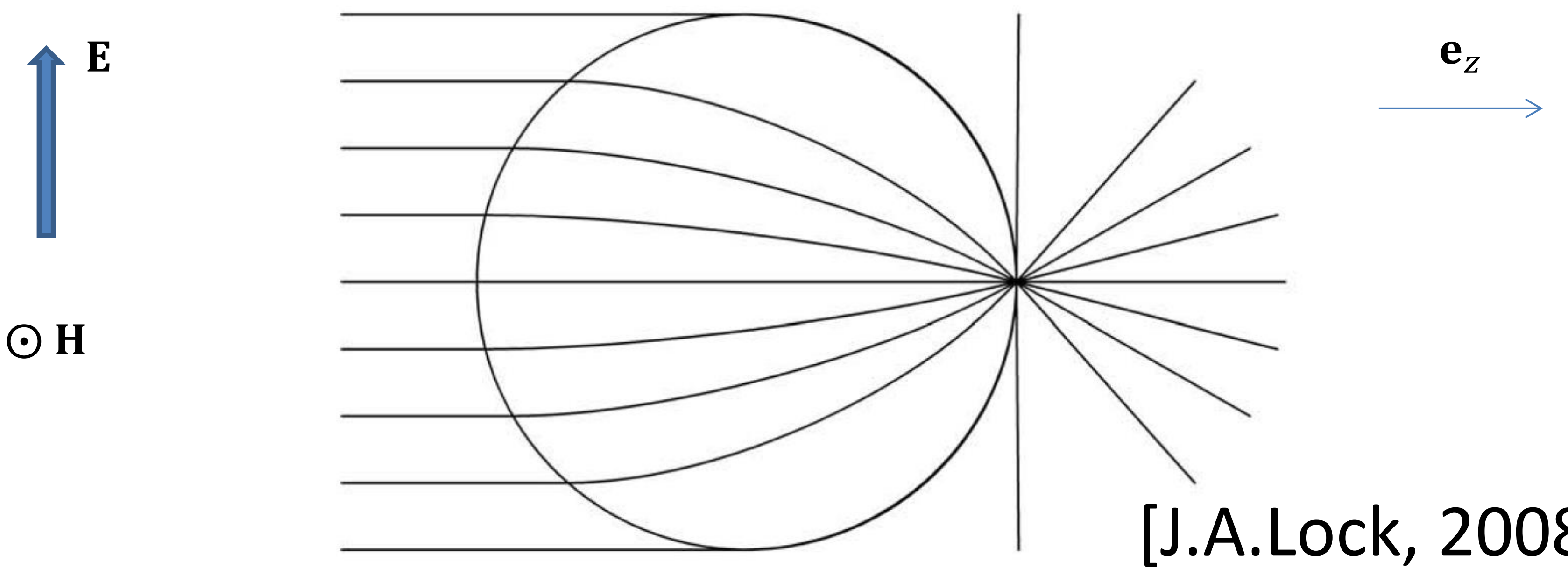
Factors u_n, v_n depend on the wave number k, the harmonic number n, and on the filling of the ball.

The class of functions $\{\varepsilon(r), \mu(r)\}$ for which the equations are simultaneously solvable in Maple'2019 is very small.

Lüneburg lens: dielectric and magnetic permeability

$$\varepsilon(r) = 2 - r^2, 0 < r < 1,$$
$$\varepsilon(r) = 1, r \geq 1;$$
$$\mu(r) = 1, r > 0.$$

For this filling the scattered field has a **focal point** on the surface of the ball.



[J.A.Lock, 2008]

Solutions of ODE for Lüneburg lens

$$\frac{d}{dr}\frac{1}{\mu(r)}\frac{du_n}{dr} + \left[k^2\varepsilon(r) - \frac{n(n+1)}{\mu(r)r^2}\right]u_n = 0:$$

$$u_n = \frac{\mathrm{WittakerM}(\frac{k}{2}, \frac{2n+1}{4}, kr^2)}{\sqrt{r}} \quad [Lock, 2008]$$

$$\frac{d}{dr}\frac{1}{\varepsilon(r)}\frac{dv_n}{dr} + \left[k^2\mu(r) - \frac{n(n+1)}{\varepsilon(r)r^2}\right]v_n = 0$$

$$v_n = r^{n+1}e^{\frac{kr^2}{2}}\mathrm{HeunC}\left(2k, \frac{2n+1}{2}, -2, k^2, -k^2 + \frac{3}{4}, \frac{r^2}{2}\right), CAS\ Maple$$

Whittaker's functions

Definition:

$$w'' + \left(-\frac{1}{4} + \frac{\chi}{z} + \frac{\frac{1}{4} - \mu^2}{z^2} \right) w = 0$$

$$\mathrm{WittakerM}(\chi, \mu, z) = z^{\mu+\frac{1}{2}} \exp\left(-\frac{z}{2}\right) {}_1F_1(\frac{1}{2} - \chi + \mu, 2\mu + 1, z)$$

$${}_1F_1(a, c, z) = 1 + \frac{a}{c}\frac{z}{1!} + \frac{a(a+1)}{c(c+1)}\frac{z^2}{2!} + \cdots$$

For Lüneburg lens:

$$\chi = \frac{k}{2}, \mu = \frac{n}{2} + \frac{1}{4}, z = kr^2, n \in \mathbb{N}.$$

Elementary Whittaker's functions for Lüneburg lens

Let be $m \in \mathbb{Z}_+$.

The elementary condition has the form:

$$\frac{n-k}{2}+\frac{3}{4}=-m \;\Leftrightarrow k=n+2m+\frac{3}{2}\Leftrightarrow n=k-2m-\frac{3}{2}$$

For $k=38.5$ we have 19 values of n for which the function $u_n(r)$ is elementary:

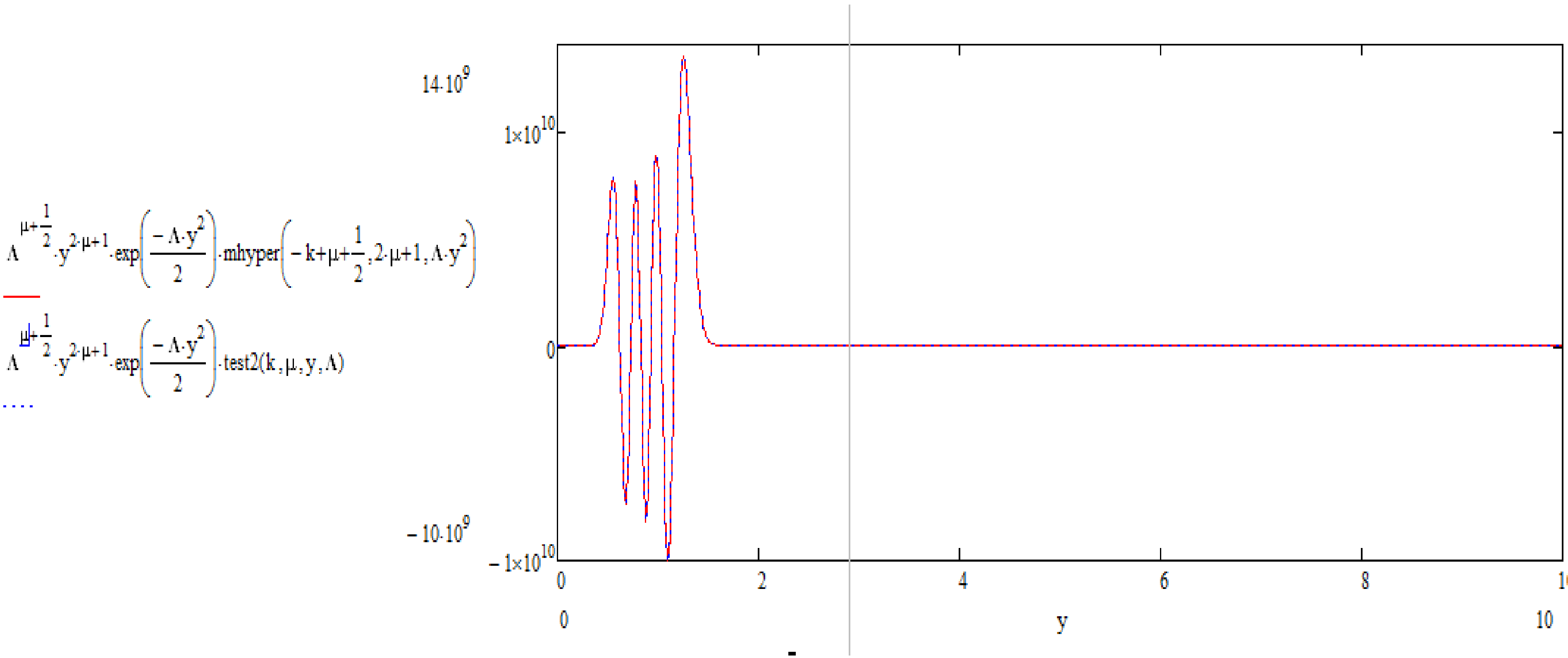
$$n=37-2m, m=\overline{0,18}$$

Elementary Whittaker's functions for Lüneburg lens: example

$$k = 38{,}5;\ n = 25$$

$$\mathrm{WittakerM}\left(\chi, \mu, kr^2\right) = \frac{334487141619119594088991 7\sqrt[4]{616}}{16384} \times r^{\frac{53}{2}} \exp\left(-\frac{77}{4} r^2\right) \times \left(\frac{2706784157}{489266055} r^{12} - \frac{492142574}{18120965} r^{10} + \frac{3195731}{59413} r^8 - \frac{16612}{3021} r^6 + \frac{1617}{53} r^4 - \frac{462}{53} r^2 + 1\right)$$

Example, $k = 38{,}5$; $n = 25$



Calculation of u_n: experiment

1. Demonstration of calculation u_n for $k = 38{,}5$ and $n = \overline{1{,}40}$, CAS MathCad. Satisfactory results alternate with unsatisfactory ones.
2. Demonstration of calculation u_n for $k = 80$ and $n = \overline{1{,}150}$, CAS MathCad. All results are unsatisfactory.

Very big values of functions!

Heun's functions for Lüneburg Lens

$$\frac{d}{dr}\frac{1}{(2-r^2)}\frac{dv_n}{dr}+\left[k^2-\frac{n(n+1)}{(2-r^2)r^2}\right]v_n=0$$

4 singular points : $r=0, r=\pm\sqrt{2}, r=\infty$

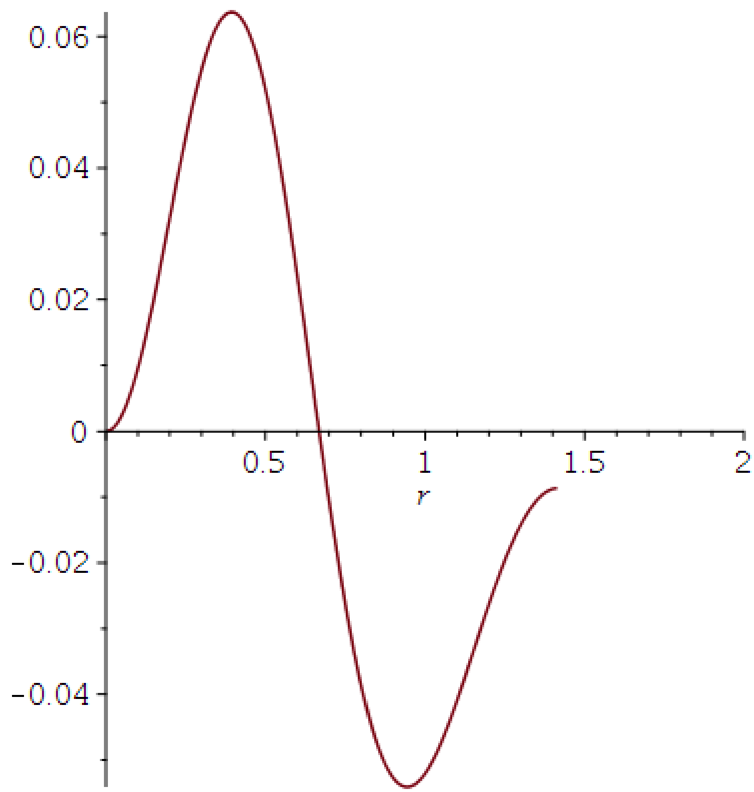
$$v_n=r^{n+1}e^{\frac{kr^2}{2}}\mathrm{HeunC}\left(2k,\frac{2n+1}{2},-2,k^2,-k^2+\frac{3}{4},\frac{r^2}{2}\right)$$

This function is bounded in the vicinity of the Fuchian singular points $r=0, r=\pm\sqrt{2}$ and is real for all real values of r.

Calculation v_n in CAS Maple 2019

Despite the reality and boundedness of the function v_n, its graph breaks off at a singular point $r = \sqrt{2}$ for all tested values k, n.

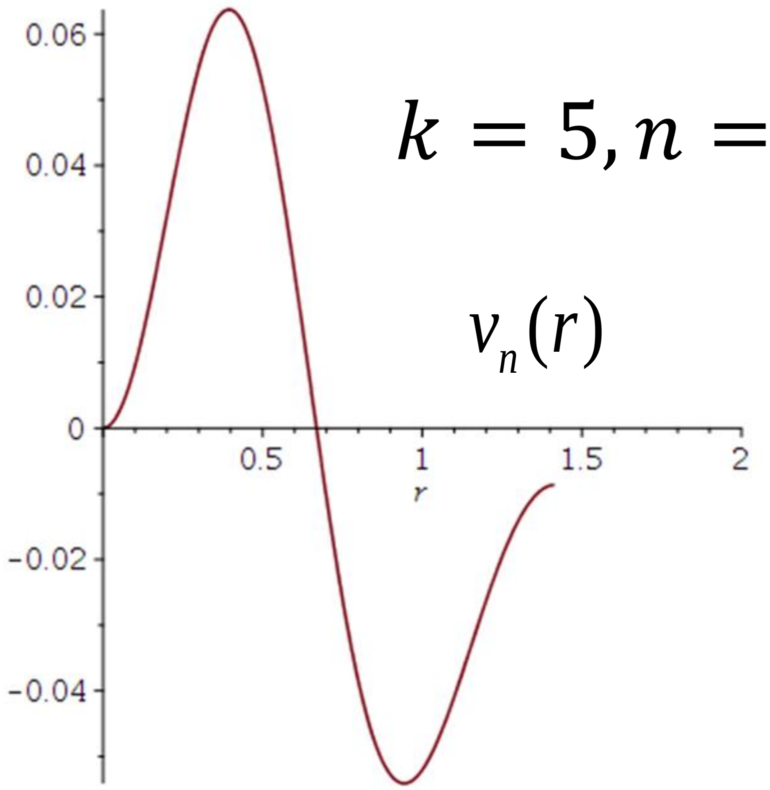
$k = 5, n = 1$

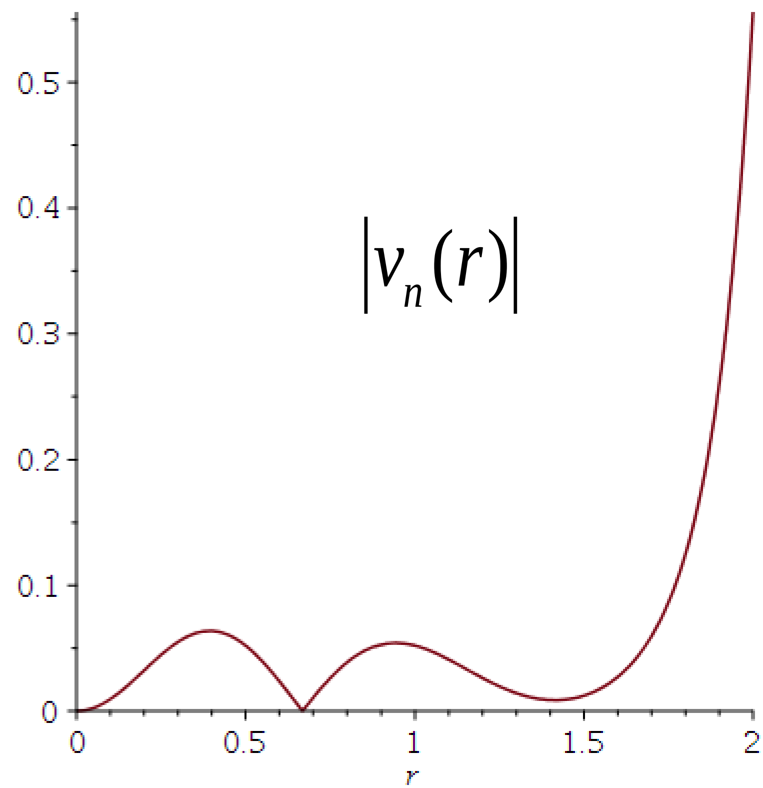


Plotting the absolute value of v_n

The "disappeared fragments" of the graphs seem to be returning from oblivion!

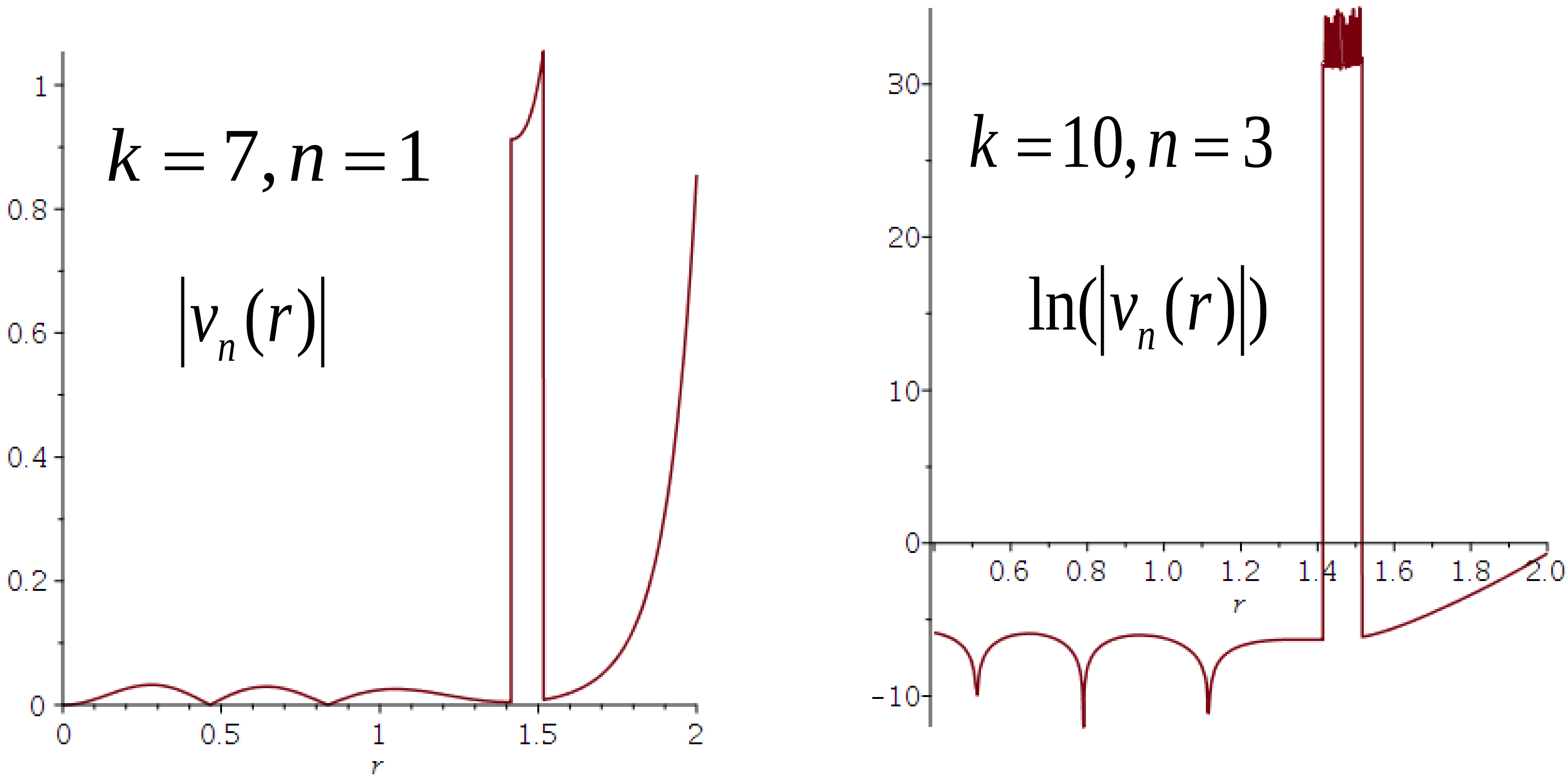
$k = 5, n = 1$





Plotting the absolute value of v_n

If $k \geq 7$, the continuation of the graph beyond the singular point $r = \sqrt{2}$ is performed incorrectly.



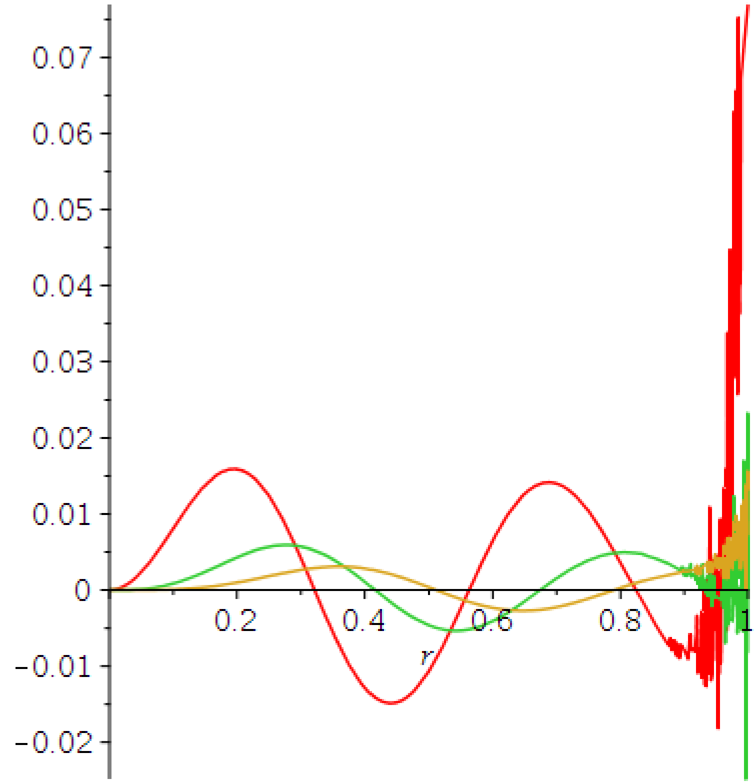
Fucshian singular point $r = \sqrt{2}$

$$v_n(r), k = 10, n = 3$$

```
> evalf(eval(V, r = sqrt(2)))
                                        Float(∞) + Float(∞) I
> evalf(eval(V, r = 1/10000 + sqrt(2)))
                                        6.790862511 10^14 - 1.781857066 10^14 I
> evalf(eval(V, r = -1/10000 + sqrt(2)))
                                        -0.001812863369
```

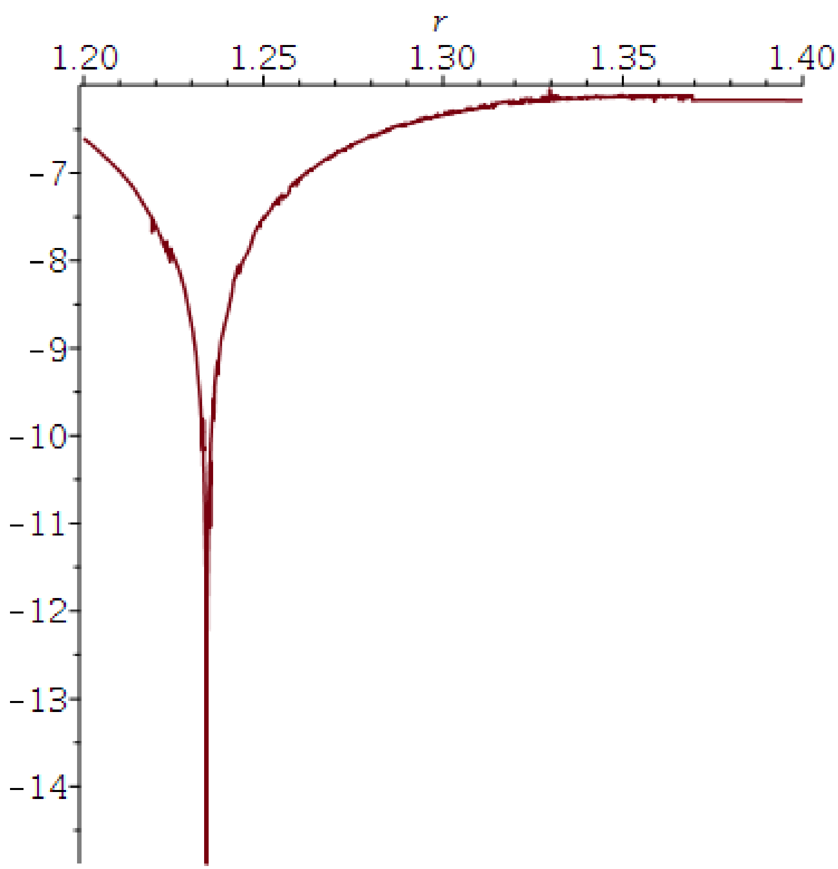
A similar picture takes place for all tested values of k, n.

Graphs of v_n in Maple 11, $k = 10, n = 1, 2, 3$



Function graphs v_n in Maple 2019

Similar problems are observed in another range of parameter k, for $k \geq 18, n \geq 1$.



$k = 18, n = 1$

$\ln(|v_n(r)|)$

“Fuchs for Sage”

To compare the results of Maple and the results from independent source we wrote the routine “Fuchs for Sage” in CAS Sage. This package allows to find solutions of linear differential equations with Fuchsian singular points in the form of Frobenius series.

$$x^2 y'' + xP(x)y' + Q(x)y = 0$$

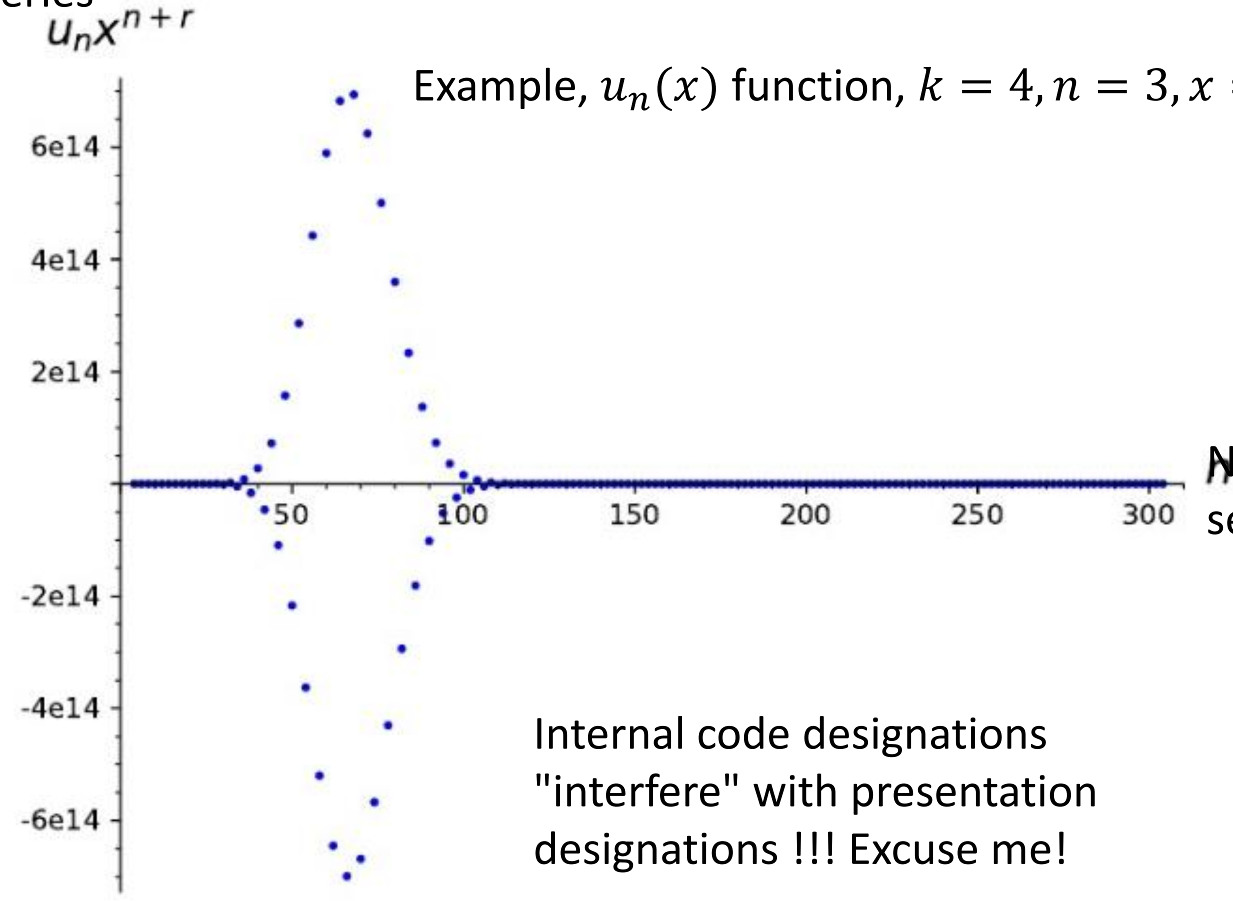
In order to eliminate the influence of round-off errors, calculations are carried out in the field of rational numbers.

Difficulty of summing the Frobenius series

The sequence of members of the Frobenius series is non-monotonic.

Example, $u_n(x)$ function, $k = 4, n = 3, x = 4$

Term of the series

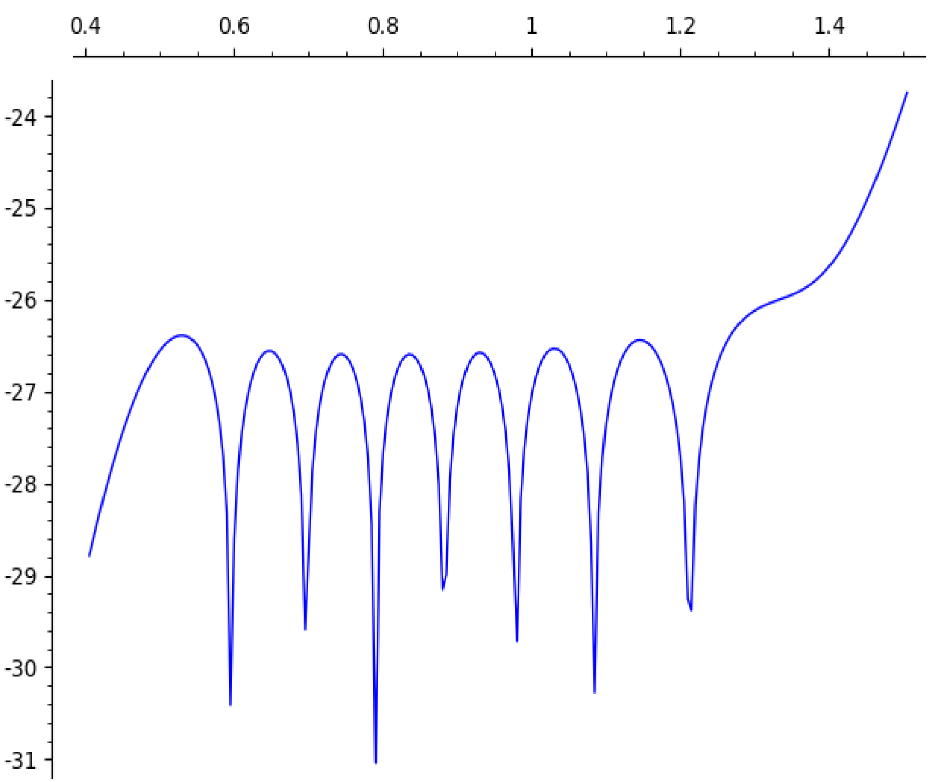


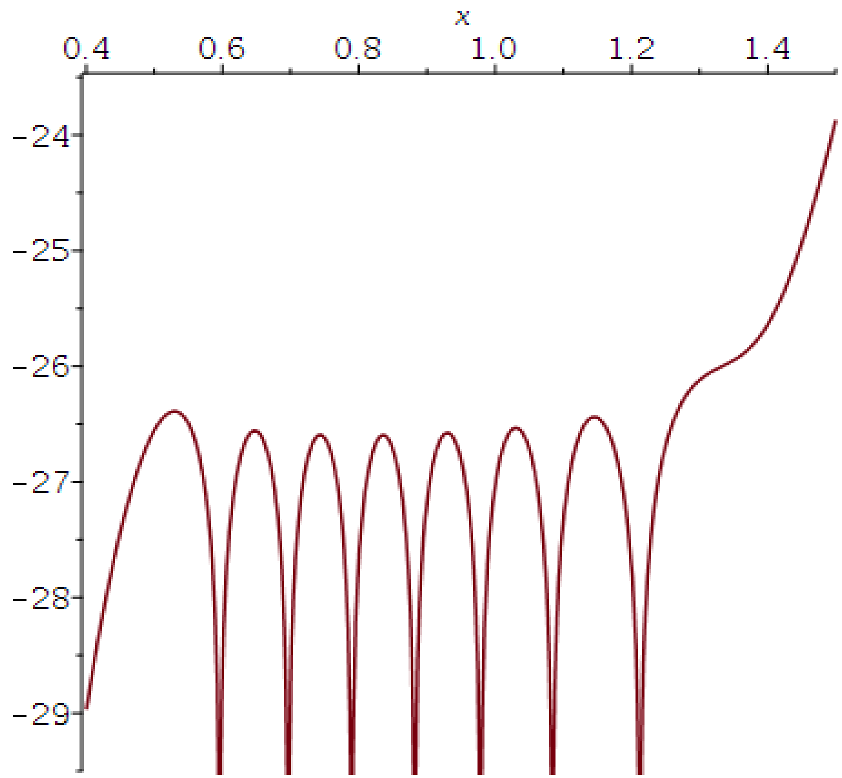
Number of series term

Internal code designations "interfere" with presentation designations !!! Excuse me!

Verification: Wittaker's functions u_n, $k = 40, n = 25$

We used 400 series terms and 221 rational points on r-axis. The graphs in the Logarithmic scale





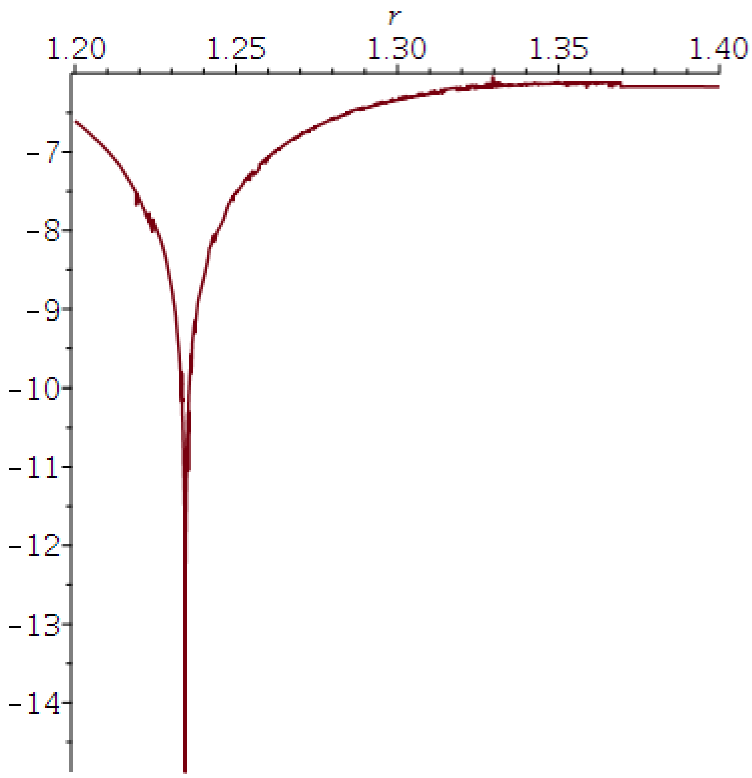
Verification: points

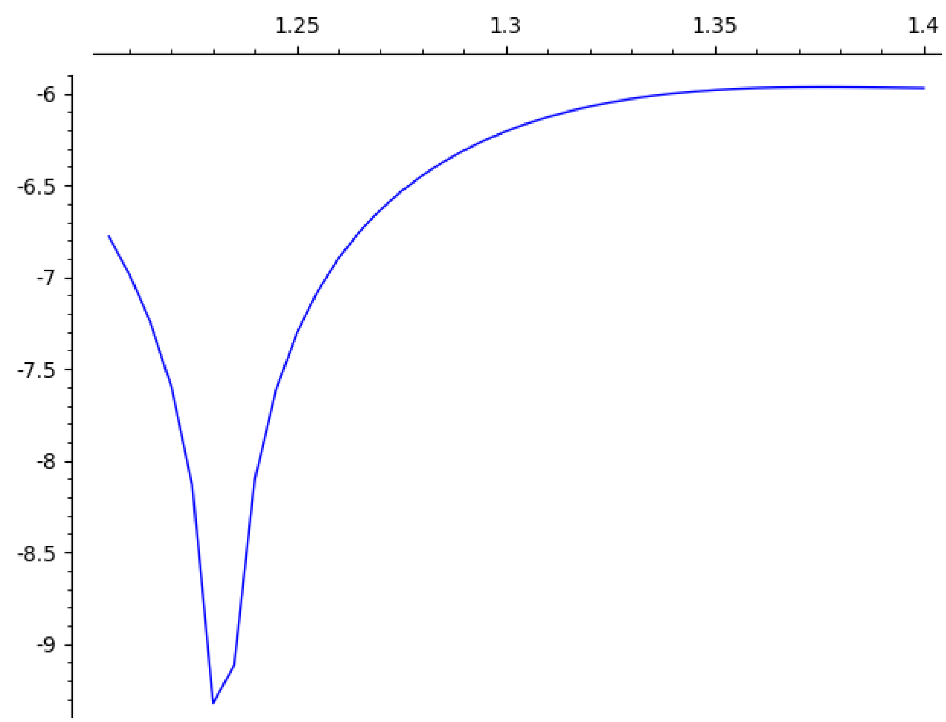
The results of assuming the Frobenius series and calculations in Maple' 2019 for Wittaker's functions $u_n, k = 10, n = 3$. Logarithmic scale.

$\boldsymbol{r}$	$\frac{1}{2}$	1	3
Number of terms in the series	50	100	500
«Fucsh for SAGE»	-7.92874739435119	-5.909705101727370	51.3761521918252
Maple' 2019	-7.928747395	-5.909705102	51.37615219

Plotting of the Heun's functions v_n

The graphs of $\ln|v_n(r)|$ for $k=18$, $n=1$ in Maple'2019 (left), and in our code (right). We used 200 series terms and 41 rational points on r-axis.





Convergence in the singular point $r = \sqrt{2}$

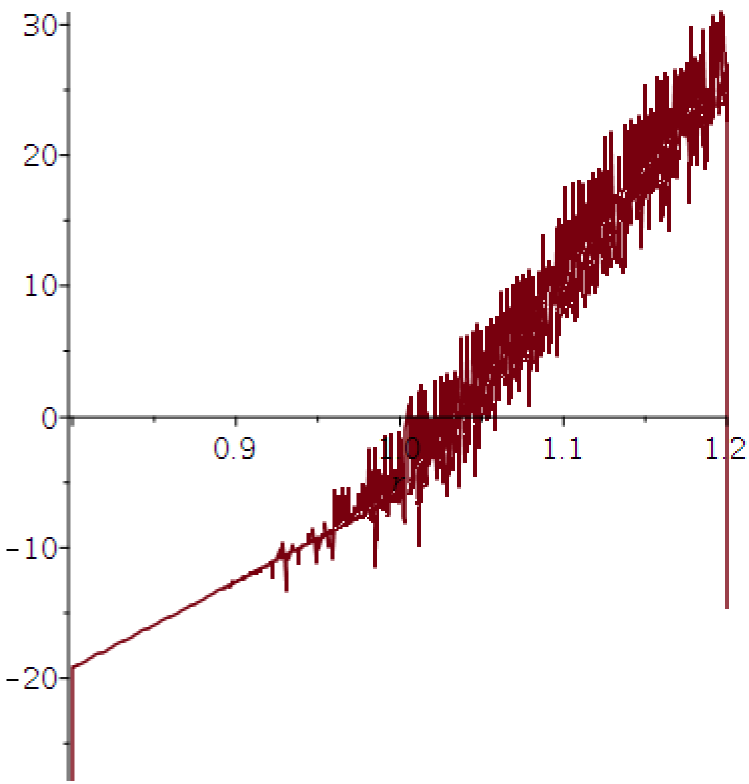
Summation of the Frobenius series **in the field of real numbers** indicates its convergence at the singular point $\sqrt{2}$. Example, $k = 10, n = 3$.

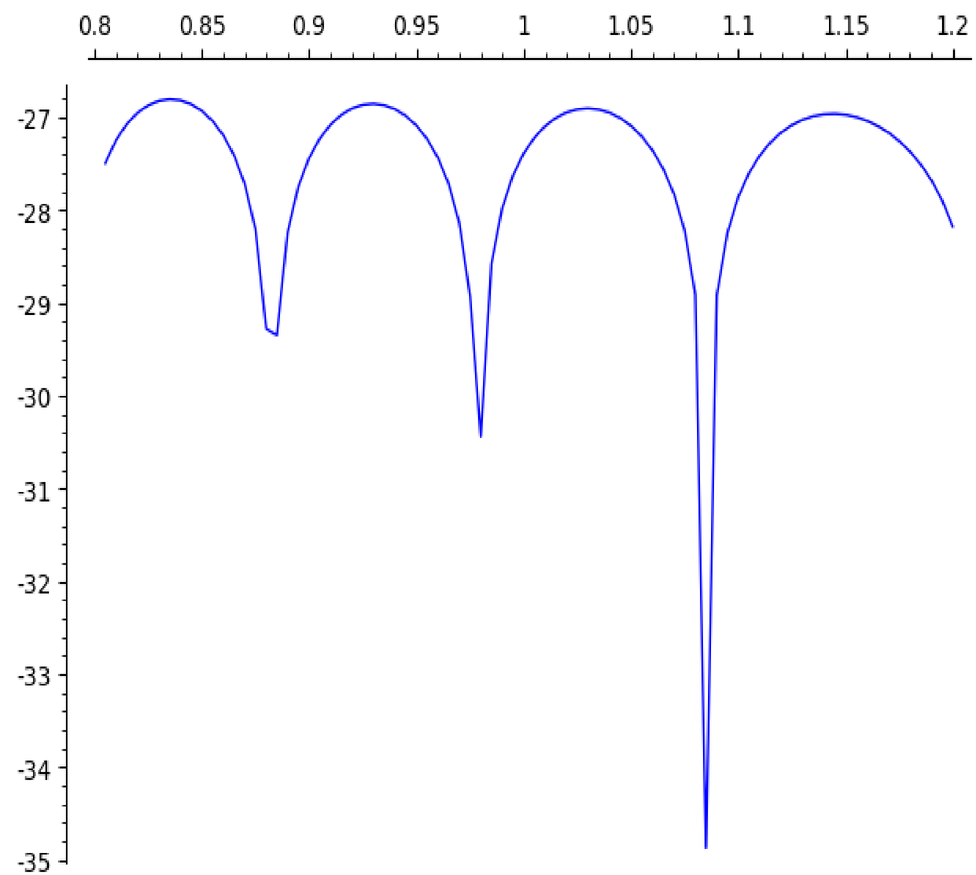
Number of terms in the series	**Partial Sum**
100	$-0.0018137250305\,2628$
300	$-0.0018128288255\,7290$
700	$-0.0018127328278\,9891$
1000	$-0.001812721658\,7255$

But this convergence is very slow compared to the convergence in the cases $r < \sqrt{2}$.

Reconstruction

The graphs of $\ln|v_n(r)|$ for $k = 40, n = 25$ in Maple'2019 (left), and in our code (right). We used 250 series terms and 82 rational points on r-axis.





Conclusions

1. Examples of erroneous calculation of special functions in Maple' 2019 are presented.

2. In CAS Sage implements an algorithm for finding these functions in the form of Frobenius series.

3. The written software package makes it possible to find solutions of linear ordinary differential equations in some cases when their solution is not expressed terms of known special functions.

Bibliography

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Thank You For Your Attention!