Multifrequency resonant conditions in Hamiltonian systems

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Abstract. The conditions on the coefficients of the characteristic polynomial of the matrix of the linearized Hamiltonian system, under which this polynomial has roots satisfying the resonance equation, are formulated. These conditions are described as roots of quasi-homogeneous polynomials defined in the coefficient space.

Introduction

Resonances play an essential role in vibrational systems. Their presence, on the one hand, leads to complex dynamics, when the energy of vibrations is "pumped" between several degrees of freedom, whose corresponding frequencies are in resonance. On the other hand, the presence of nontrivial solutions of the resonance equation allows to write additional formal first integrals and, as a consequence, allows to analyze the stability of the equilibrium position or to integrate asymptotically the system of equations of motion reduced to the normal form.

Consider an analytic Hamiltonian system

$$\dot{\mathbf{x}} = \frac{\partial H}{\partial \mathbf{y}}, \quad \dot{\mathbf{y}} = -\frac{\partial H}{\partial \mathbf{x}} \tag{1}$$

with n degrees of freedom, where $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, near the equilibrium position

$$\mathbf{x} = \mathbf{y} = 0.$$

The Hamilton function $H(\mathbf{x}, \mathbf{y})$ expands into a convergent power series

$$H(\mathbf{x}, \mathbf{y}) = \sum H_{\mathbf{pq}} \mathbf{x}^{\mathbf{p}} \mathbf{y}^{\mathbf{q}}$$

with constant coefficients $H_{\mathbf{pq}}$, where $\mathbf{p}, \mathbf{q} \geq 0$, $|\mathbf{p}| + |\mathbf{q}| \geq 2$.

Introduce a phase vector $\mathbf{z} = (\mathbf{x}, \mathbf{y}) \in \mathbb{R}^{2n}(\mathbb{C}^{2n})$. Then the linear part of the system (1) can be written in the form

$$\dot{\mathbf{z}} = B\mathbf{z}, \quad B = \frac{1}{2}\begin{pmatrix} \dfrac{\partial^2 H}{\partial \mathbf{y}\partial \mathbf{x}} & \dfrac{\partial^2 H}{\partial \mathbf{y}\partial \mathbf{y}} \\ -\dfrac{\partial^2 H}{\partial \mathbf{x}\partial \mathbf{x}} & -\dfrac{\partial^2 H}{\partial \mathbf{x}\partial \mathbf{y}} \end{pmatrix}\Bigg|_{\mathbf{x}=\mathbf{y}=0} \tag{2}$$

Let $\lambda_1, \ldots, \lambda_{2n}$ be the eigenvalues of matrix B, which can be reordered as follows $\lambda_{j+n} = -\lambda_j$, $j = 1, \ldots, n$. Denote by vector $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_n)$ the set of basic eigenvalues of the system (2). For a Hamiltonian system, the characteristic polynomial $\check{f}(\lambda)$ is the polynomial of even powers of λ. Let us call the polynomial $f(\mu) \stackrel{\text{def}}{=} \check{f}(\lambda)$, where $\mu = \lambda^2$, as *semi-characteristic*:

$$f(\mu) = \mu^n + a_1\mu^{n-1} + a_2\mu^{n-2} + \cdots + a_{n-1}\mu + a_n. \tag{3}$$

According to Theorem 12 in [1, § 12] in the case of semi-simple eigenvalues there exists a canonical formal transformation that reduces the Hamiltonian system (1) to its *normal form*

$$\dot{\mathbf{u}} = \partial h/\partial \mathbf{v}, \quad \dot{\mathbf{v}} = -\partial h/\partial \mathbf{u},$$

given by the normalized Hamiltonian $h(\mathbf{u}, \mathbf{v})$

$$h(\mathbf{u}, \mathbf{v}) = \sum_{j=1}^{n} \lambda_j u_j v_j + \sum h_{\mathbf{pq}} \mathbf{u}^{\mathbf{p}} \mathbf{v}^{\mathbf{q}} \tag{4}$$

containing only the resonant terms $h_{\mathbf{pq}} \mathbf{u}^{\mathbf{p}} \mathbf{v}^{\mathbf{q}}$ satisfying the *resonant equation*

$$\langle \mathbf{p} - \mathbf{q}, \boldsymbol{\lambda} \rangle = 0. \tag{5}$$

Here $\langle \mathbf{p}, \boldsymbol{\lambda} \rangle = \sum_j = 1^n p_j \lambda_j$ is the scalar product.

The resonant equation (5) has two kinds of solutions, which correspond to two kinds of resonant terms in the normal form (4):

1. *Secular terms* of the form $h_{\mathbf{pp}} \mathbf{u}^{\mathbf{p}} \mathbf{v}^{\mathbf{p}}$, which are always present in the Hamiltonian normal form because of the special structure of the matrix B of the linearized system (2)
2. *Strictly resonant terms*, which correspond to nontrivial integer solutions of the equation

$$\langle \mathbf{p}, \boldsymbol{\lambda} \rangle = 0. \tag{6}$$

To investigate the formal stability of the equilibrium position of the Hamiltonian system it is necessary to perform a normalization procedure and then apply Bruno's theorem [3], the condition of which requires the absence of third- and fourth-order resonances. The conditions on the coefficients of the polynomial (3) for two-frequency resonances are effectively formulated in terms of q-discriminants [4].

The problem is the following: formulate conditions on the coefficients a_j, $j = 1, \ldots, n$, of the semi-characteristic polynomial $f(\mu)$ of degree $n = 3$ and

$n = 4$, under which the multifrequency resonance of multiplicity 1 of order 3 or order 4 takes place.

1. Conditions for a system with three degrees of freedom

The only multifrequency resonance of multiplicity 1 of order 3 corresponds to the case where the algebraic sum of all three basic eigenvalues λ_j, $j = 1, 2, 3$, is equal to zero. Then in terms of roots μ_j of the polynomial (3) this condition is written as

$$\mu_1 = A \pm 2C, \text{ where } A = \mu_2 + \mu_3, C^2 = \mu_2\mu_3. \tag{7}$$

Considering the condition (7) as a polynomial ideal, we compute the Gröbner elimination basis that excludes the quantities A and C, and obtain the condition on roots:

$$\sum_{j=1}^{3} \mu_j^2 - 2 \sum_{1=j<k}^{3} \mu_j\mu_k = \sigma_1^2(\mu) - 4\sigma_2(\mu), \tag{8}$$

where $\sigma_k(\mu)$ are elementary symmetric polynomials for which $\sigma_k(\mu) = (-1)^k a_k$. Then the condition on the coefficients of the polynomial (3) takes the form

$$a_1^2 - 4a_2 = 0. \tag{9}$$

The condition for the existence of multifrequency resonance of multiplicity 1 of order 4 is equivalent to the case of the algebraic sum of $2\lambda_1$, λ_2, λ_3 equals to zero. Repeating the above calculations, we get a condition on the coefficients of the polynomial (3) in the form

$$16\, a_1^6 - 264\, a_1^4 a_2 + 36\, a_1^3 a_3 + 1425\, a_1^2 a_2^2 - 630\, a_1 a_2 a_3 - 2500\, a_2^3 + 9261\, a_3^2 = 0. \tag{10}$$

The conditions (9) and (10) are algebraic varieties in the coefficient space of the polynomial (3) for $n = 3$, and their left-hand sides are quasi-homogeneous polynomials from coefficients a_j, $j = 1, 2, 3$. By methods of power geometry one can obtain a polynomial parametrization of the variety (10):

$$a_1 = 2v\,(37t - 35)\,, a_2 = \left(456337t^2 - 7666t + 721\right) v^2,$$
$$a_3 = 36\,(71t + 2)\,(5 - 249t)^2\, v^3.$$

Note that many works on oscillation theory consider multifrequency resonances of multiplicities 2 and higher. For example, a resonance of order 4 in the form of commensurable frequencies 2 : 1 : 1 has multiplicity 2 and is defined using the conditions on two-frequency resonances. The 1 : 1 commensurability is determined by the discriminant variety $\mathcal{D}(f)$ of dimension 2, the 2 : 1 commensurability is determined by the resonant variety $\mathcal{R}_4(f)$ of dimension 2, and their intersection gives the submanifold of dimension 1 on which the above resonance takes place. At the same time, the condition (10) for the existence of a three-frequency resonance of multiplicity 1 defines a variety of dimension 2, i. e. it is a more general condition.

2. Conditions for a system with four degrees of freedom

In this case, the situations of three-frequency and four-frequency resonances should already be considered separately. We have two three-frequency resonances of orders 3 and 4, and one four-frequency resonance of order 4.

The condition on roots (8) of three-frequency resonance of order 3 for a polynomial of degree 4 must be satisfied for some one triplet of roots μ_k, $k = 1, \dots, 4$. Then we make a product of four factors of the form (7) for each triplet and add it to the ideal composed of polynomials of the form $a_k - \sigma_k$, $k = 1, \dots, 4$. Using the Gröbner elimination basis, we exclude the values of μ_k and obtain a condition on the coefficients of the form

$$-4\,a_1^5a_3 + a_1^4a_2^2 + 4\,a_1^4a_4 + 34\,a_1^3a_2a_3 - 8\,a_1^2a_2^3 - 30\,a_1^2a_2a_4 - 27\,a_1^2a_3^2 - 72\,a_1a_{2^2a_3} + 16\,a_2^4 - 54\,a_1a_3a_4 + 72\,a_2^2a_4 + 108\,a_2a_3^2 + 81\,a_4^2 = 0.$$

We can do the same to determine the condition on the roots of the polynomial in the presence of a three-frequency resonance of order 4 for a polynomial of degree 4. The resulting polynomial turns out to be very cumbersome (it contains 153 monomials) and is not given here.

Finally, let us indicate the condition on the coefficients of the polynomial (3) of the fourth order which, if satisfied, leads to a four-frequency resonance of order 4:

$$a_1^4 - 8\,a_1^2a_2 + 16\,a_2^2 - 64\,a_4 = 0.$$

References

[1] A. D. Bruno. Analytical form of differential equations (II). *Trans. Moscow Math. Soc.*, 26:199–239, 1972. = Trudy Moskov. Mat. Obsc. 25 (1971) 119-262 (in Russian).

[2] A. D. Bruno. *The Restricted 3–body Problem: Plane Periodic Orbits.* Walter de Gruyter, Berlin, 1994. = Nauka, Moscow, 1990. 296 p. (in Russian).

[3] A. D. Bruno. Formal stability of Hamiltonian systems. *Math. Notes*, 1(3):216–219, 1967.

[4] A. B. Batkhin. Parameterization of a set determined by the generalized discriminant of a polynomial. *Programming and Computer Software*, 44(2):75–85, 2018. https://doi.org/10.1134/S0361768818020032.

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