Multifrequency resonant conditions in Hamiltonian systems

Alexander Bruno, Alexander Batkhin, Zafar Khaydarov
abruno@keldysh.ru batkhin@gmail.com
zafarxx@gmail.com

Keldysh Institute of Applied Mathematics
Moscow Institute of Physics and Technology
Moscow, Russia
Samarkand State University named after Sh. Rashidov, Samarkand, Uzbekistan

PCA '2022, May 2 – 7, 2022

Talk outlook

1. Types of stability and problem setting
2. Conditions for a system with three degrees of freedom
3. Conditions for a system with four degrees of freedom
4. Conclusion

Abstract

Resonances play an essential role in vibrational systems. Their presence, on the one hand, leads to complex dynamics, when the energy of vibrations is "pumped" between several degrees of freedom, whose corresponding frequencies are in resonance. On the other hand, the presence of nontrivial solutions of the resonance equation allows to write additional formal first integrals and, as a consequence, allows to analyze the stability of the equilibrium position or to integrate asymptotically the system of equations of motion reduced to the normal form

Types of stability

There are three types of stability for theoretical-mechanical problems (see [Bruno, 2020]):

Types of stability

There are three types of stability for theoretical-mechanical problems (see [Bruno, 2020]):

- mathematical stability (Lyapunov stability)

There are three types of stability for theoretical-mechanical problems (see [Bruno, 2020]):

- mathematical stability (Lyapunov stability)
- formal stability (Moser stability)

Types of stability

There are three types of stability for theoretical-mechanical problems (see [Bruno, 2020]):

- mathematical stability (Lyapunov stability)
- formal stability (Moser stability)
- practical stability

Types of stability

There are three types of stability for theoretical-mechanical problems (see [Bruno, 2020]):

- mathematical stability (Lyapunov stability)
- formal stability (Moser stability)
- practical stability

Lyapunov stability is the most rigorous and guarantees uniform boundedness of solutions over an infinite interval of time with respect to the set of perturbations by initial conditions and parametres

Types of stability

There are three types of stability for theoretical-mechanical problems (see [Bruno, 2020]):

- mathematical stability (Lyapunov stability)
- formal stability (Moser stability)
- practical stability

Lyapunov stability is the most rigorous and guarantees uniform boundedness of solutions over an infinite interval of time with respect to the set of perturbations by initial conditions and parametres

Moser stability is weaker, but ensures a slower rate of scattering of trajectories than the power growth with an arbitrary positive exponent

Types of stability

There are three types of stability for theoretical-mechanical problems (see [Bruno, 2020]):

- mathematical stability (Lyapunov stability)
- formal stability (Moser stability)
- practical stability

Lyapunov stability is the most rigorous and guarantees uniform boundedness of solutions over an infinite interval of time with respect to the set of perturbations by initial conditions and parametres

Moser stability is weaker, but ensures a slower rate of scattering of trajectories than the power growth with an arbitrary positive exponent

Practical stability means only bounded solutions on a finite interval with respect to a set of disturbing factors

Stability of stationary point (1)

In the generic case, an analytic time-independent Hamiltonian function $H(\mathbf{z})$ in the vicinity of the ***stationary point*** (SP), coinciding with the origin, is expanded into a convergent series of homogeneous polynomials H_k of degree k of its phase variables $\mathbf{z} = (\mathbf{x}, \mathbf{y})$

$$H(\mathbf{z}; \mathbf{P}) = \sum_{k=2}^{\infty} H_k(\mathbf{z}; \mathbf{P}), \qquad (1)$$

where $\mathbf{P}$ is a vector of parameters

It is known that stability in the first approximation can be determined only for the case when the quadratic form $H_2(\mathbf{z})$ is sign-defined (Lagrange-Dirichlet theorem)

Stability of stationary point (2)

If the number of degrees of freedom is not more than two

- Stability is determined by the Arnold–Moser theorem in the absence of resonances of order four or less, which requires normalizing H to order four;
- For resonances of order less than four, the stability conditions were derived in the works of A. P. Markeev and A. G. Sokolsky (see [Markeev, 1978])

When the number of degrees of freedom is more than two, stability for most initial conditions is determined by Arnold's theorem (see, for example, [Markeev, 1978]).

From a practical point of view, a weaker than Lyapunov stability is quite sufficient, *formal stability* proposed by J. Moser.

Stability set of the linear Hamiltonian system (1)

The series (1) starts with the quadratic Hamiltonian $H_2(\mathbf{z};\mathbf{P})$ defining the local dynamics near the SP. The behavior of the phase flow in the first approximation is described by a linear Hamiltonian system

$$\dot{\mathbf{z}}(t) = B(\mathbf{P})\mathbf{z}, \quad B(\mathbf{P}) = \frac{1}{2}J\frac{\partial^2 H_2(\mathbf{P})}{\partial\mathbf{z}\partial\mathbf{z}} \tag{2}$$

Let us recall here the main properties of a linear Hamiltonian system

1. If λ_j is an eigenvalue of the matrix B, then $-\lambda_j$ is also its eigenvalue. All eigenvalues λ_j, $j = 1,\ldots,2n$, of the matrix B can be reordered in such a way that $\lambda_{j+n} = -\lambda_j$, $j = 1,\ldots,n$. Denote by vector $\boldsymbol{\lambda} = (\lambda_1,\ldots,\lambda_n)$ the set of ***basic eigenvalues***

Stability set of the linear Hamiltonian system (2)

2. The characteristic polynomial $\check{f}(\lambda)$ of the matrix B contains only even powers of λ, so it is a polynomial in $\mu = \lambda^2$. The following polynomial is called ***semi-characteristic***

$$f(\mu) = \sum_{k=0}^{n} f_{n-k}(\mathbf{P})\mu^k, \quad f_0 \equiv 1 \tag{3}$$

3. If $\mathrm{Re}\,\lambda_j \neq 0$ for some j, i.e., the SP is hyperbolic, then it is unstable
4. If all $\mathrm{Re}\,\lambda_j = 0$, the behavior of the phase flow in its vicinity can only be obtained by taking into account the nonlinear terms. Usually this is performed using KAM-theory, but here such study is performed using the Hamiltonian normal form (NF)

Stability set of the linear Hamiltonian system (3)

Definition 1.

The stability set Σ of the linear system (2) is the set of all values of parameters $\mathbf{P} \in \Pi$ for which the SP $\mathbf{z} = 0$ is Lyapunov stable

In terms of roots of a semi-characteristic polynomial (3), the condition of stability of the SP is given by

Theorem 2 ([Batkhin, Bruno, Varin, 2012]).

The SP $\mathbf{z} = 0$ *of the linear Hamiltonian system* (2) *is Lyapunov-stable if and only if*

- *All the roots* μ_k *of the semi-characteristic polynomial* (3) *are real and non-positive*
- *All elementary divisors of the matrix* B *are simple*

Definition 3.

The SP $\mathbf{z} = 0$ of a system with Hamilton function $H(\mathbf{z})$ is ***formally stable*** if there exists a possibly divergent power series $G(\mathbf{z})$ which is a formal positively defined first integral $\{G, H\} = 0$, where $\{\cdot, \cdot\}$ is the Poisson bracket

Some results on formal stability (2)

In [Bruno, Batkhin, 2021] a schematic description of the method for studying the formal stability of the SP was given. This method is based on the following key results:

- normal form of the Hamiltonian at the SP
- Bruno's Theorem on formal stability
- q-analog of the classical elimination theory

under the following assumptions:

- Number of degrees of freedom more than two
- The quadratic form $H_2(\mathbf{z})$ in the expansion (1) is nondegenerate and not sign-defined
- Hamiltonian function $H(\mathbf{z})$ smoothly depends on the parameter vector $\mathbf{P}$

Some results on formal stability (3)

In the absence of strong resonances between eigenvalues of a linearized Hamiltonian system in the neighborhood of the SP, the condition for its formal stability is formulated by the Bruno theorem

Theorem 4 (Bruno [Bruno, 1972]).

For any analytic Hamiltonian $H(\mathbf{z})$ *there exists a formal canonical transformation* $\mathbf{z} \to \mathbf{w} = (\mathbf{u}, \mathbf{v})$ *that*

$$H(\mathbf{z}) = h(\mathbf{w}) = \sum h_{\mathbf{pq}} \mathbf{u}^{\mathbf{p}} \mathbf{v}^{\mathbf{q}},$$

where $\mathbf{p}, \mathbf{q} \in \mathbb{Z}^n$, $\mathbf{p}, \mathbf{q} \geqslant 0$ *and constants* $h_{\mathbf{pq}} \neq 0$ *only for resonant term* $\langle \pm p - \mathbf{q}, \boldsymbol{\lambda} \rangle = 0$

Some results on formal stability (4)

Condition A_k^n

Resonant equation

$$\langle \mathbf{p}, \boldsymbol{\lambda} \rangle = 0$$

has no integer solutions $\mathbf{p}$ with $|\mathbf{p}| \stackrel{\text{def}}{=} |p_1| + \cdots + |p_n| \leqslant k$

Let the condition A_4^n take place, i.e., $\langle \mathbf{K}, \boldsymbol{\lambda} \rangle \neq 0$ for $\mathbf{K} \in \mathbb{Z}^n, \quad 0 < \|\mathbf{K}\| \leqslant 4$, then it is known that there exists an analytic canonical transformation $(\mathbf{x}, \mathbf{y}) \to (\boldsymbol{\rho}, \boldsymbol{\varphi})$ such that the new Hamiltonian g has the form

$$g(\boldsymbol{\rho}, \boldsymbol{\varphi}) = g_2(\boldsymbol{\rho}) + g_4(\boldsymbol{\rho}) + r(\boldsymbol{\rho}, \boldsymbol{\varphi}),$$

where $g_2(\boldsymbol{\rho}) = \langle \boldsymbol{\lambda}, \boldsymbol{\rho} \rangle$, $g_4(\boldsymbol{\rho}) = \langle C\boldsymbol{\rho}, \boldsymbol{\rho} \rangle$, $C = [c_{ij}]_{i,j=1}^n$, and $r(\boldsymbol{\rho}, \boldsymbol{\varphi})$ is a convergent power series of variables $(\boldsymbol{\rho}, \boldsymbol{\varphi})$ of degree three or higher in $\boldsymbol{\rho}$

Some results on formal stability (5)

Theorem 5 (Bruno [Bruno, 1967]).

If Condition A_k^n takes place and for any nonzero integer vectors $\mathbf{K}$ of ortant $k_i \geqslant 0$, $i = 1, \ldots, n$, which is a solution to the equation

$$\langle \mathbf{K}, \boldsymbol{\lambda} \rangle = 0, \tag{4}$$

quadratic form $\langle \mathbf{C}\mathbf{K}, \mathbf{K} \rangle \neq 0$ at $\boldsymbol{\lambda} \neq 0$, then the SP $\mathbf{z} = 0$ of the Hamilton system is formally stable

Note that the condition (4) of the Bruno theorem is equivalent to the fact that the semi-algebraic system $g_2(\boldsymbol{\rho}) = g_4(\boldsymbol{\rho}) = 0, \boldsymbol{\rho} \geqslant 0$ is incompatible.

Thus, to apply the Bruno theorem on formal stability, it is necessary to find the boundaries of regions in the parameter space Π defined by resonance sets

Following [Bruno, 1994, Ch. I, § 3] we define

Definition 6.

Multiplicity of resonance $\mathfrak{k}$ is the number of linearly independent solutions $\mathbf{p} \in \mathbb{Z}^n$ to the resonant equation $\langle \mathbf{p}, \boldsymbol{\lambda} \rangle = 0$

Order of resonance is equal to $\mathfrak{q} = \min |\mathbf{p}|$ by $\mathbf{p} \in \mathbb{Z}^n$, $\mathbf{p} \neq 0, \langle \mathbf{p}, \boldsymbol{\lambda} \rangle = 0$

If the solution to the resonant equation contains only two eigenvalues, then such resonance is called a ***two-frequency resonance***, if more than two, then it is called a ***multi-frequency resonance***

Problem setting (2)

To investigate the formal stability of the equilibrium position of the Hamiltonian system it is necessary to perform a normalization procedure and then apply Bruno's theorem, the condition of which requires the absence of third- and fourth-order resonances. The conditions on the coefficients of the semi-characteristic polynomial (3) for two-frequency resonances are effectively formulated in terms of q-discriminants [Batkhin, 2018]

The problem is the following:

Formulate conditions on the coefficients a_j, $j = 1, \ldots, n$, of the semi-characteristic polynomial $f(\mu)$ of degree $n = 3$ and $n = 4$, under which the multi-frequency resonance of multiplicity 1 of order 3 or order 4 takes place

More detailed: for resonance of

order 2: $\mathbf{p} = (1,1)$ – case of multiple roots, described by discriminant $D(f) = 0$

order 3: for two-frequency case $\mathbf{p} = (2,1)$, described by q-discriminant $D_q(f) = 0$, $q = 4$

order 4: for two-frequency case $\mathbf{p} = (3,1)$, described by q-discriminant $D_q(f) = 0$, $q = 9$

multi-frequency case: order 3, $\mathbf{p} = (1,1,1)$, order 4, $\mathbf{p} = (2,1,1)$, or $\mathbf{p} = (1,1,1,1)$

Two-frequency resonances (1)

Let consider Hamiltonian system with 3 *degrees of freedom* (DOF). Its semi-characteristic polynomial is cubic

$$f_3(\mu) = \mu^3 + a_1\mu^2 + a_2\mu + a_3 \tag{5}$$

There are three two-frequency resonances of order no greater then four:

$$1:1, \quad 2:1, \quad 3:1$$

Resonance $1:1$

Condition on coefficients is given by discriminant $D(f_3) = 0$:

$$D(f_3) \stackrel{\text{def}}{=} -4a_1^3a_3 + a_1^2a_2^2 + 18a_1a_2a_3 - 4a_2^3 - 27a_3^2 = 0$$

which defines a ruled surface with directrix as a screwed cubic curve

Two-frequency resonances (2)

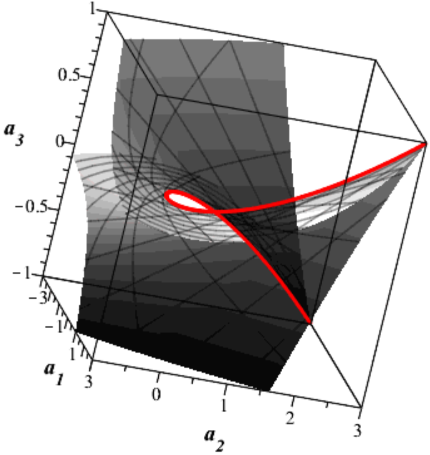

Figure 1: Discriminant variety for cubic polynomial f_3

Two-frequency resonances (3)

Discriminant surface $\mathcal{D}(f_3) : \{D(f_3) = 0\}$ divides the space into two parts with different types of roots:

- one part corresponds to the case when all roots are real
- another part correspond to the case of one real and a pair of complex roots

Two-frequency resonances (4)

Resonances $p:1$, $p = 2{,}3$

Condition on coefficients is given by q-discriminant $D_q(f_3) = 0$:

$$\begin{aligned} D_q(f_3) = {} & q^2\left(1+q\right)^2 a_1^3 a_3 - q^3 a_1^2 a_2^2 - \\ & - q\left(1+q+q^2\right)\left(1+4\,q+q^2\right) a_1 a_2 a_3 + \\ & + q^2\left(1+q\right)^2 a_2^3 + \left(1+q+q^2\right)^3 a_3^2 = 0 \end{aligned}$$

which also defines a ruled surface with directrix as a screwed cubic curve. Here q should be given as p^2, i.e. $q = 4$ and $q = 9$

Two-frequency resonances (5)

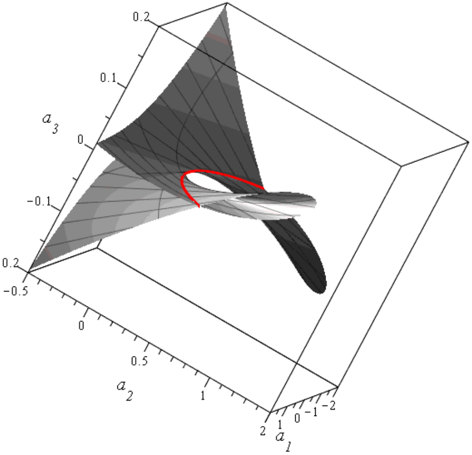

Figure 2: Resonance variety for cubic polynomial f_3

Two-frequency resonances (6)

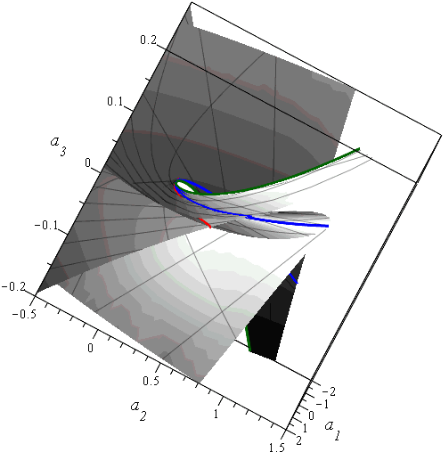

Figure 3: Mutual position of discriminant and resonance varieties for f_3

Multi-frequency resonance (1)

The only multi-frequency resonance of multiplicity 1 of order 3 corresponds to the case where the algebraic sum of all three basic eigenvalues λ_j, $j = 1,2,3$, is equal to zero:

$$\lambda_1 = \pm\lambda_2 \pm \lambda_3$$

Then in terms of roots μ_j of the polynomial (5) this condition is written as

$$\sqrt{\mu_1} = \pm\sqrt{\mu_2} \pm \sqrt{\mu_3},$$

which is equivalent to the equation

$$(\mu_1 - \sigma_1(\mu_2, \mu_3))^2 - 4\sigma_2(\mu_2, \mu_3) = \sigma_1^2(\boldsymbol{\mu}) - \sigma_2(\boldsymbol{\mu}) = 0$$

where $\boldsymbol{\mu} = (\mu_1, \mu_2, \mu_3)$, $\sigma_k(\boldsymbol{\mu})$ are elementary symmetric polynomials for which $\sigma_k(\boldsymbol{\mu}) = (-1)^k a_k$

Multi-frequency resonance (2)

Then the condition on the coefficients of the polynomial (5) takes the form

$$\mathrm{Cond}_3^{(1,1,1)} \stackrel{\mathrm{def}}{=} a_1^2 - 4a_2 = 0, \tag{6}$$

which describes a parabolic cylinder. It intersects with $\mathcal{D}(f_3)$ along the curve $\{a_2 = a_1^2/4, a_3 = a_1^3/54\}$

Multi-frequency resonance (3)

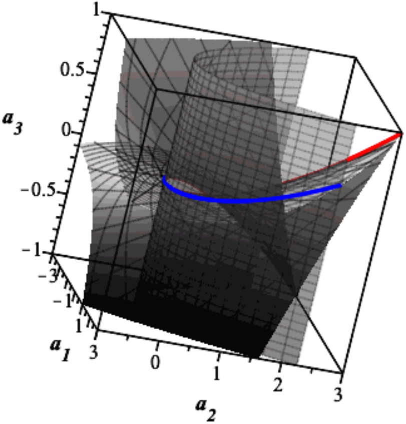

Figure 4: Mutual position of discriminant and $\mathrm{Cond}_3^{(1,1,1)}$

Multi-frequency resonance (4)

The condition for the existence of multifrequency resonance of multiplicity 1 of order 4 is equivalent to the case of the algebraic sum of $2\lambda_1$, λ_2, λ_3 equals to zero:

$$2\lambda_1 = \pm\lambda_2 \pm \lambda_3 \text{ or } \sqrt{4\mu_1} = \pm\sqrt{\mu_2} \pm \sqrt{\mu_3}$$

Last condition is equivalent to a polynomial

$$16\mu_1^2 - 8\mu_1\mu_2 - 8\mu_3\mu_1 + \mu_2^2 - 2\mu_3\mu_2 + \mu_3^2 = 0$$

Condition on the coefficients of the polynomial (5) takes the form

$$\begin{aligned} \mathrm{Cond}_3^{(2,1,1)} \stackrel{\mathrm{def}}{=} 16\,a_1^6 - 264\,a_1^4 a_2 + 36\,a_1^3 a_3 + 1425\,a_1^2 a_2^2 - \\ -630\,a_1 a_2 a_3 - 2500\,a_2^3 + 9261\,a_3^2 = 0 \end{aligned} \tag{7}$$

The conditions (6) and (7) are algebraic varieties in the coefficient space of the polynomial (5) for $n = 3$, and their left-hand sides are quasi-homogeneous polynomials from coefficients a_j, $j = 1,2,3$. By methods of power geometry one can obtain a polynomial parametrization of the variety (7):

$$a_1 = 2v\,(37t - 35)\,, a_2 = \left(456337t^2 - 7666t + 721\right) v^2,$$
$$a_3 = 36\,(71t + 2)\,(5 - 249t)^2\, v^3$$

Multi-frequency resonance (6)

Equation (7) defines a surface with self-intersecting along the screwed cubic with parametrization

$$a_1 = 35t, \quad a_2 = 259t^2, a_3 = 225t$$

There is a resonance of multiplicity 2 and order 9 with three pairs of commensurable roots: $\lambda_2 : \lambda_1 = 3 : 1$, $\lambda_3 : \lambda_1 = 5 : 1$ and $\lambda_3 : \lambda_2 = 5 : 3$, i.e. $2\lambda_1 = \lambda_3 - \lambda_2$

Multi-frequency resonance (7)

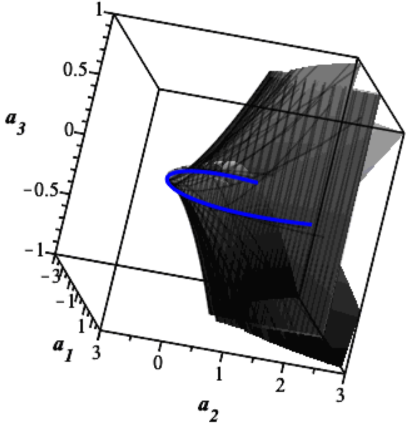

Figure 5: Surface of resonant condition $\mathrm{Cond}_3^{(2,1,1)}$

Remark

Note that many works on oscillation theory consider multi-frequency resonances of multiplicities 2 and higher. For example, a resonance of order 4 in the form of commensurable frequencies $2:1:1$ has multiplicity 2 and is defined using the conditions on two-frequency resonances. The $1:1$ commensurability is determined by the discriminant variety $\mathcal{D}(f)$ of dimension 2, the $2:1$ commensurability is determined by the resonant variety $\mathcal{D}_4(f)$ of dimension 2, and their intersection gives the submanifold of dimension 1 on which the above resonance takes place. At the same time, the condition (7) for the existence of a three-frequency resonance of multiplicity 1 defines a variety of dimension 2, i. e. it is a more general condition

Here we consider Hamiltonian system with 4 DOF. Its semi-characteristic polynomial is quartic

$$f_4(\mu) = \mu^4 + a_1\mu^3 + a_2\mu^2 + a_3\mu + a_4 \tag{8}$$

In this case, the situations of three-frequency and four-frequency resonances should already be considered separately. We have

- two three-frequency resonances of orders 3 and 4: $\mathbf{p} = (1,1,1)$, $\mathbf{p} = (2,1,1)$, and
- one four-frequency resonance of order 4: $\mathbf{p} = (1,1,1,1)$

Three-frequency resonance (1)

Resonance of order 3

Any of three roots should satisfy condition $\lambda_i = \pm\lambda_j \pm \lambda_k$ for any triplet (i, j, k), $i \neq j \neq k$
In terms of μ_j it takes form

$$\prod_{(i,j,k)} \left[\sigma_1^2(\mu_i, \mu_j, \mu_k) - 4\sigma_2(\mu_i, \mu_j, \mu_k)\right] = 0$$

over all triplets (i, j, k), $i, j, k = 1{,}2{,}3{,}4$

In coefficients a_i, $i = 1{,}2{,}3{,}4$ of polynomial (8) it can be written as

$$\begin{aligned}\mathrm{Cond}_4^{(1,1,1)} \stackrel{\mathrm{def}}{=} -4a_1^5a_3 + a_1^4a_2^2 + 4a_1^4a_4 + 34a_1^3a_2a_3 - 8a_1^2a_2^3 - \\ -30a_1^2a_2a_4 - 27a_1^2a_3^2 - 72a_1a_2^2a_3 + 16a_2^4 - \\ -54a_1a_3a_4 + 72a_2^2a_4 + 108a_2a_3^2 + 81a_4^2 = 0\end{aligned}$$

Resonance of order 4

Any of three roots should satisfy condition $2\lambda_i = \pm\lambda_j \pm \lambda_k$ for any triplet (i,j,k), $i \neq j \neq k$

In μ_i this condition takes form

$$\prod_{(i,j,k)} \left[16\mu_i^2 - 8\mu_i\mu_j - 8\mu_k\mu_i + \mu_j^2 - 2\mu_k\mu_j + \mu_k^2\right] = 0$$

over all triplets (i,j,k), $i,j,k = 1,2,3,4$

Condition on coefficients of $f_4(\mu)$ contains 153 monomials and can not be written here

Resonance of order 4

All four roots should satisfy condition $\lambda_1 = \pm\lambda_2 \pm \lambda_3 \pm \lambda_4$ or in μ_j, $j = 1,2,3,4$, it is equivalent to

$$\left((\mu_1 - \sigma_1(\boldsymbol{\mu}'))^2 - 4\sigma_2(\boldsymbol{\mu}')\right)^2 - 64\sigma_3(\boldsymbol{\mu}')\mu_1 = 0$$

Here $\boldsymbol{\mu} = (\mu_2, \mu_3, \mu_4)$

Condition on coefficients of $f_4(\mu)$ takes form

$$\mathrm{Cond}_4^{(1,1,1,1)} \stackrel{\mathrm{def}}{=} a_1^4 - 8\,a_1^2 a_2 + 16\,a_2^2 - 64\,a_4 = 0$$

1. Types of stability and problem setting

2. Conditions for a system with three degrees of freedom

3. Conditions for a system with four degrees of freedom

4. Conclusion

Conclusion

- Increasing the degree of semi-characteristic polynomial leads to very complex expression of resonant conditions

Conclusion

- Increasing the degree of semi-characteristic polynomial leads to very complex expression of resonant conditions
- We suppose that the resonant multifrequency conditions can admit polynomial parametrization

Conclusion

- Increasing the degree of semi-characteristic polynomial leads to very complex expression of resonant conditions
- We suppose that the resonant multifrequency conditions can admit polynomial parametrization
- Presence of multi-frequency resonance in Hamiltonian system of multiplicity 1 makes possible to find additional first formal integrals and makes Hamiltonian system formally integrable

References (1)

- *Bruno A. D.* Formal stability of Hamiltonian systems. // Mathematical Notes. 1967. Vol. 1, no. 3. P. 216–219.
- *Bruno A. D.* Analytical form of differential equations (II). // Trans. Moscow Math. Soc. 1972. Vol. 26. P. 199–239.
- *Markeev A. P.* Libration Points in Celestial Mechanics and Cosmodynamics. Moscow: Nauka, 1978. (in Russian).
- *Bruno A. D.* The Restricted 3–body Problem: Plane Periodic Orbits. Berlin: Walter de Gruyter, 1994.
- *Batkhin A. B.*, *Bruno A. D.*, *Varin V. P.* Stability sets of multiparameter Hamiltonian systems. // Journal of Applied Mathematics and Mechanics. 2012. Vol. 76, no. 1. P. 56–92. ISSN 0021-8928. DOI: 10.1016/j.jappmathmech.2012.03.006. URL: http://www.sciencedirect.com/science/article/pii/S0021892812000329.

References (2)

Batkhin A. B. Parameterization of a Set Determined by the Generalized Discriminant of a Polynomial. // Programming and Computer Software. 2018. Vol. 44, no. 2. P. 75–85. DOI: https://doi.org/10.1134/S0361768818020032.

Bruno A. D. On types of stability in Hamiltonian systems. // Preprints of KIAM. 2020. No. 21. P. 24. DOI: 10.20948/prepr-2020-21. (in Russian).

Bruno A. D., *Batkhin A. B.* Survey of eight modern methods of Hamiltonian mechanics. // Axioms. 2021. Vol. 10, no. 4. DOI: 10.3390/axioms10040293. URL: https://www.mdpi.com/2075-1680/10/4/293.