The generalized arithmetic-geometric mean

Semjon Adlaj
http://SemjonAdlaj.com/

Computing Center of the Russian Academy of Sciences

This work was supported by the Russian Foundation for Basic Research (Project 19-29-14141)

Polynomial Computer Algebra annual conference

Euler International Mathematical Institute
Saint Petersburg, Russia
May 02, 2022

The GAGM defined

The modified arithmetic-geometric sequence is the recursively defined triple sequence

$$x_{n+1} := \frac{x_n + y_n}{2}, \;\; y_{n+1} := z_n + \sqrt{(x_n - z_n)(y_n - z_n)}, \;\; z_{n+1} := z_n - \sqrt{(x_n - z_n)(y_n - z_n)}.$$

Given such a sequence $\{x_n, \; y_n, \; z_n\}_{n=0}^{\infty}$, we introduce (another) recursively defined sequence of (single-valued) parametric functions:

$$u_{n+1} = u_{n+1}(t) = u_{n+1}(t, c, x_0, y_0, z_0) := \frac{c_n \, u_n - y_{n+1} \, z_{n+1}}{c_n + u_n - 2 \, z_n}, \;\; c_n := u_n(c),$$

where c is a fixed real parameter and the function u_0 is (naturally) presumed to coincide with the identity function: $u_0(t) = t$. We proceed to defining the functions

$$v_n = v_n(t) = v_n(t, a, c, x_0, y_0, z_0) := \frac{t - a_n}{t - c_n}, \;\; a_n := u_n(a),$$

$$w_n = w_n(t) = w_n(t, b, a, c, x_0, y_0, z_0) := \frac{v_n(t)}{v_n(b_n)}, \;\; b_n := u_n(b),$$

where a and b are (also) fixed real parameters distinct from c and each other.

Define the generalized arithmetic-geometric mean (GAGM) of two (strictly) positive numbers x and y, for a given pairwise distinct real parameters a, b and c, as the (common) limit of the sequence $\{\xi_n := w_n(x_n)\}_{n=0}^{\infty}$ and the sequence $\{\eta_n := w_n(y_n)\}_{n=0}^{\infty}$ with $x_0 = x$, $y_0 = y$ and $z_0 = 0$.

We might then extend the domain of the parameters a, b and c to include the point at (complex) infinity, so that a, b and c might be regarded as elements of the extended real line $\mathbb{R} \cup \infty$. However, we shall always require the parameter c to lie (strictly) outside the closed interval $[x, \; y]$.

The invariance of the GAGM under the action of linear functions

Given a linear function $l(t) = \lambda\,(t-\mu)$, $\{\lambda \neq 0,\ \mu\} \subset \mathbb{R}$, we define an action of the function l upon the sextuple sequence as $l \cdot \{x_n,\ y_n,\ z_n,\ a_n,\ b_n,\ c_n\}_{n=0}^{\infty} := \{l(x_n),\ l(y_n),\ l(z_n),\ l(a_n),\ l(b_n),\ l(c_n)\}_{n=0}^{\infty}$, thereby inducing an action upon the sequence $\{w_n\}_{n=0}^{\infty}$, which we shall denote by $l \cdot \{w_n\}_{n=0}^{\infty} := \{l \cdot w_n\}_{n=0}^{\infty}$, where $l \cdot w_n$ is the transformation mapping the ordered triple $(l(a_n),\ l(b_n),\ l(c_n))$ to the ordered triple $(0,\ 1,\ \infty)$. One might then verify that such an action is well defined, and, furthermore, neither the sequence $\{\xi_n\}_{n=0}^{\infty}$ nor $\{\eta_n\}_{n=0}^{\infty}$ is altered by this action, that is,

$$\xi_n = l \cdot w_n\big(l(x_n)\big) = w_n(x_n),\ \ \eta_n = l \cdot w_n\big(l(y_n)\big) = w_n(y_n).$$

In particular, The homogeneity degree of GAGM is zero (unlike the AGM and MAGM which are homogeneous of degree one), and we might exploit this property to extend the domain of GAGM, for fixed parameters, to include (strictly) negative values of the arguments x and y. At each iteration, we might ensure the positivity of the product $(x_n - z_n)(y_n - z_n)$, before taking its square root, via acting upon the sextuple sequence (at the required step whenever necessary) by the (constant) function -1.

We shall denote with the same letter N three functions, which we shall nevertheless distinguish by the (total) number of their arguments. The invariance of the GAGM under the action of linear functions upon the sextuple sequence implies that four initial arguments suffice to determine the GAGM, so we designate $N(x,\ a, b, c)$ to denote the GAGM of 1 and x for parameters a, b and c. Moreover, the expression

$$\Big(\frac{(b-a)\,N\,(x,\ a, b, c)}{b-c} - 1\Big)/(c-a),$$

while seemingly dependent upon four arguments x, a, b and c, has x and c as its only "true" arguments. It actually depends neither upon a nor upon b. Consequently, we might define a two-variable function

$$N(x,\ c) := N(x,\ \infty, c+1, c),$$

and employ it in order to alternatively express the preceding four-variable function as

$$N\,(x,\ a, b, c) = \frac{b-c}{b-a}\Big((c-a)\,N\,(x,\ c) + 1\Big).$$

The GAGM: Extension via inversion

The preceding formula extends not only to the case $c = 0$ but, as well, to the case $c = \infty$. In these two (dual) cases the GAGM "degenerates" to a (shifted) MAGM:

$$N\left(x,\ a, b, 0\right) = \frac{b}{a-b}\left(a\,N\left(\frac{1}{x}\right) - 1\right),\ \ N\left(x,\ a, b, \infty\right) = \frac{N(x) - a}{b-a},$$

where the (one-variable) function $N(x)$ is the modified arithmetic-geometric mean of 1 and x. The equivalence of the latter two equations reflects a special (limit) case of the relation

$$N\left(x,\ a, b, c\right) = N\left(\frac{1}{x},\ \frac{1}{a}, \frac{1}{b}, \frac{1}{c}\right).$$

The latter relation suggests that the defining equality of the function $N(x,\ c)$ might be substituted with the equality

$$N(x,\ c) = N\left(\frac{1}{x},\ 0,\ \frac{1}{c+1},\ \frac{1}{c}\right),$$

which is suitable for explicit calculation, and is extendable to the case $c = 0$ as

$$N(x,\ 0) = N\left(\frac{1}{x}\right),$$

but, unlike the four-variable function, the two-variable function remains undefined for $c = \infty$.

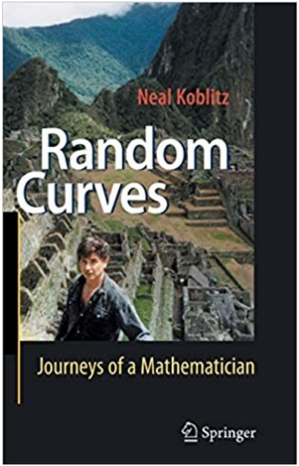
"Two, correction requiring, common misconceptions"

Upon teaching calculus, Neal Koblitz insisted that $\ln(x)$ is not an antiderivative of $1/x$, unless it is "corrected" to be $\ln|x|$.

So, once being "enlightened" with his advice, one might proceed to (correctly) calculate the definite integral:

$$\int_{-e}^{-1} \frac{dx}{x} = \ln|x|\Big|_{-e}^{-1} = \ln 1 - \ln e = -1.$$

N. Koblitz, no less often, "enlightened naive and confused people", telling them:



"elliptic curves have nothing to do with ellipses"!

The AGM as a GAGM

The AGM, being expressible as a GAGM,

$$M(\beta) = N\left(\beta,\, \frac{1}{\beta},\, 0,\, \beta - i\sqrt{1-\beta^2},\, 1,\, \beta + i\sqrt{1-\beta^2}\right),$$

that is, the AGM of β and 1 coincides with the GAGM of β and $1/\beta$ for parameters $\beta - i\sqrt{1-\beta^2}$, 1 and $\beta + i\sqrt{1-\beta^2}$,[1] enables a swift (quadratically convergent) calculation of the ratio $N(\beta^2)/M(\beta)$, via an iterative procedure, requiring at each step (aside from basic arithmetic operations) a single square root extraction.

[1] Thus, if β, $1/\beta$ and 0 are regarded to be the values of the three half-period of an essential elliptic function, then $\beta - i\sqrt{1-\beta^2}$, 1 and $\beta + i\sqrt{1-\beta^2}$ are the values at three (out of six) quarter-periods of the same (Galois) function.

Acknowledgments and a dedication

This work was supported by the Russian Foundation for Basic Research (Project № 19-29-14141, headed by Sergei Pozdniakov).

It is dedicated to a lasting memory of V.V. Shevchenko (1953-2022).
So, we shall bring to attention a few excerpts from two late letters of V.V. Shevchenko.

Из электронного письма 12 ноября 2021 года: Удар должен быть безошибочным, наверняка. К директору ФСБ, председателю СК, Генеральному прокурору ... Параллельно необходимо выйти на центральную прессу, лучше телевидение ... Тогда должно получиться. Эта банда липовых академиков пустила разветвлённые корни, выкорчёвывать их надо умело и основательно.

Из электронного письма 1 декабря 2021 года: В 1991 году А.А. Дородницын написал иронический «гимн новой российской академической науки», одно из четверостиший которого звучит так: «Это бой наш последний, В нём мы всех победим, И в академии российской, Науке места не дадим». Наблюдаемое нами, увы, показывает, что события разворачиваются именно так. Среди адресатов данной переписки вижу действующего Президента РАН А.М. Сергеева. Обращаюсь к нему: «Александр Михайлович! Ответ на наше предложение декабря 2015 года по сей день не получен. Будьте любезны ответить. Чтобы нам не пришлось писать «донос на гетмана злодея царю Петру от Кочубея». Не доводите до того, чтобы в соответствии с одним из предсказаний М.В. Ковальчука РАН погибла, как Римская Империя.
В.В. Шевченко (в представлении не нуждаюсь).

Several, chronologically ordered, references

[1] *Gauß' wissenschaftliches Tagebuch 1796-1814*. Mathematische Annalen, Springer Berlin Heidelberg, **57**: 1-34.

[2] Mathematics Stack Exchange. *Modified Arithmetic-Geometric Mean*. Answered by Francois Lamarche on Oct 5, 2016 and edited on Dec 9, 2019. Available at https://math.stackexchange.com/questions/391382/modified-arithmetic-geometric-mean

[3] Pozdniakov S. *Using Math Partner Environment in Algorithmic Mathematics Course*. **14**th International Conference on Polynomial Computer Algebra. Euler International Mathematical Institute, St. Petersburg, April 19–24, 2021. Available at https://pca-pdmi.ru/2021/files/41/pca_example_Pozdniakov.pdf

[4] Malaschonok G. & Seliverstov A. *New Features in MathPartner 2021*. Computer tools in education, 2021, № **3**: 29–40. Available at http://www.ipo.spb.ru/journal/content/2286/Новые%20функции%20в%20MathPartner%202021.pdf

[5] Malaschonok G. & Seliverstov A. *Calculation of integrals in MathPartner*. Discrete and Continuous Models and Applied Computational Science, 2021, Vol. 29, № **4**: 337-346. Available at https://doi.org/10.22363/2658-4670-2021-29-4-337-346

[6] Adlaj S. *An explicit procedure for calculating the perimeter of an ellipse*. **22**nd Workshop on Computer Algebra in memory of Professor Vladimir Gerdt, Dubna, Russia, May 24–25, 2021. Available at https://indico-hlit.jinr.ru/event/187/contributions/1769/attachments/543/931/SAdlaj.pdf