The generalized arithmetic-geometric mean

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Abstract. The generalized arithmetic-geometric mean (GAGM) is the "canonical" generalization of the arithmetic-geometric mean (AGM) which was discovered by Gauss to enable most efficient calculation of complete elliptic integrals. It thereby provides the framework necessary for a contemporary Computer Algebra System (CAS), where complete elliptic integrals of all kinds are to be exactly evaluated and robustly, swiftly calculated.

The introduction of the arithmetic-geometric mean (AGM) signified (as Gauss recorded in his diary on May 30, 1799) the emergence of a "new era of analysis" [1]. Yet, over two more centuries were required before the great significance of Gauss' discovery was understood. The concept of the modified arithmetic-geometric mean (MAGM), discussed in [2], clarified the link between the complete elliptic integral of two kinds (the second with the first). The Computer Algebra System (CAS) "MathPartner", which implementation was discussed in [3], incorporates both the AGM and the MAGM, as told in [4, 5].

The GAGM might be regarded as the concept linking to each other complete elliptic integral of all (three) kinds. The procedure for calculating the perimeter of an ellipse via the GAGM, presented in [6], emphasized that the calculation of the complete elliptic integral of the second kind did not necessarily require separate calculations of the AGM and the MAGM. We also emphasized the Gauss-Euler algorithm as the algorithm (with no other namings for this algorithm to be ever justified, contrary to claims made in [7]) underlying fast algorithms for calculating the constant π. Thus, further incorporating the GAGM in CAS "MathPartner" provides the framework necessary for exact and robust calculations of complete elliptic integrals of all kinds, including the third kind, which is notorious to many contemporary CAS for being too prone to erroneous calculations. The errors being unavoidable whenever the multivaludness of (path-dependent) elliptic integrals is not properly addressed. The exploration of the algebraic properties of the GAGM provides the necessary tools for matching its multivaludness with corresponding multivaludness of complete elliptic integrals, with the Galois elliptic function, as defined in [8], providing the basis for such exploration.

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