On the computation of Abelian differential of the third kind

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Symbolic integration in XIX century

- 1835, Liouville. Integration of an expression like

$$p(x)e^q(x), \quad p, q \in \mathbb{C}[x]$$

in elementary functions.

- 1888, Ptaszycki. Integration of the Abelian integrals, that is an expressions like

$$R(x, y), \quad f(x, y) = 0,\ f \in \mathbb{C}[x, y],\ R \in \mathbb{C}(x, y),$$

in elementary functions.

- 1875, Weierstrass. Theory of the Abelian integrals, including the reduction to normal form.

The exploration of the classic heritage

- First successes of computer algebra systems (CAS) are primarily due to the fact that back in the 1960s, it was possible to create algorithms for the symbolic integration of elementary functions, based on Liouville work [Moses].
- In the 1980s, algorithms for integration of Abelian integrals in elementary functions were developed, but the final set of developed algorithms are too complicated to be implemented in CAS [Bronstein, Devenport, Trager].

Publication of Weierstrass lecture

Weierstrass lectures ware published by Hettner and Knoblauch only in 1902, that is after publishing of popular works like Backer's monograph. The conventional wisdom about Weierstrass's theory is based on his early work and speculation.
In 1870s, Weierstrass developed a purely algebaric and constructive approach to Abelian integrals, in particular he completely revised his theory of elliptic functions [Pokrovsky, 1900].

Weierstrass lecture

In 1980s, Polubarinova-Kochina noticed that Weierstrass's approach was constructive. However, there are no examples of using the algorithms in the published text of lectures.
Modern authors see power series and analytic functions in lectures and do not see the purely algebaric theory of the fundamental function (Hauptfuktion). Furthermore, they do not see the duality principle and its main consequence — identity connecting integrals of the three kind (Art).
This identity is a brilliant trick, Pokrovsky in 1900 was unable to construct an elegant theory of elliptic functions since he did not find a corresponding lecture during his trip to Berlin.

QQbar

A characteristic feature of Weierstrass' approach is the use a lot of irrational numbers, the algorithm for determining which is either described in the text, or more or less obvious.
Now, the Sage system has a built-in implementation of the field of algebraic numbers QQbar, added by Carl Witty in 2007, so in theory the algorithms from the Lectures can be implemented as written.

Our aim

We decided to consider this direct implementation of the algorithms and evaluate the difficulties that arise, return to the subject of our talk at PCA'2018.

Refs.:

1. Yu Ying, M. D. Malykh, L. A. Sevastianov. On symbolic integration of algebraic functions. 2020. Journal of Symbolic Computation. DOI: 10.1016/j.jsc.2020.09.002
2. Yu Ying, E. A. Ayryan, M. D. Malykh, L. A. Sevastianov. On the computation of fundamental functions and Abelian differentials of the third kind. 2021. ArXiv:2105.07875

Abelian differentials of the first kind

Let polynomial $f \in \mathbb{Q}[x, y]$ define an algebraic curve C of the order r on the projective plane xy over the field $\mathbb{C}$. Let for simplicity this curve have no singular points.

Definition

A differential of the form udx, $u \in \mathbb{C}(x, y)$, having no singular points on the curve f is called a differential of the first kind.

Problem

Given a polynomial $f \in \mathbb{Q}[x, y]$, find a non-constant rational function $u \in \overline{\mathbb{C}}(x, y)$ such that udx is a differential of the first kind.

Solution of the Problem

The absence of finite singular points makes one seek the solution in the form

$$\frac{E(x,y)dx}{f_y(x,y)}, \quad E \in \mathbb{C}[x,y],$$

and the absence of singular points at infinity indicates the fact that the order of the polynomial E cannot exceed $r-3$. Since no limitations should be imposed on the coefficients of this polynomial, the set of differentials of the first kind has the dimension

$$p = \frac{(r-1)(r-2)}{2},$$

which is called a genus (Rang) of the curve. For the basis of this space one can take differentials with the coefficients form the field $\mathbb{Q}$, rather than from its algebraic closure. Therefore, when constructing differentials of the first kind it is possible and necessary to work over the field $\mathbb{Q}$.

Implementation in CAS

Algorithms for calculating a basis for the space of differentials of the first kind for planar curves, including those having singular points, have been proposed both in classical books and in present-day papers [Hoeij].
At present they are implemented in Maple system (AlgCurves, CASA) and partially in Sage.

Abelian differentials of the third kind

Definition

A differential of the form udx, $u \in \mathbb{C}(x, y)$ is called a differential of the third kind, if it has two singular points, namely, poles of the first order (x_1, y_1) and (x_2, y_2) with residues 1 and -1.

Problem

Given an indecomposable polynomial $f \in \mathbb{Q}[x, y]$, defining a projective curve C, and two points (x_1, y_1) and (x_2, y_2) on this curve, and $x_1, x_2, y_1, y_2 \in \overline{\mathbb{Q}}$. It is required to construct a non-constant rational function $u \in \mathbb{C}(x, y)$ such that udx is a differential of the third kind with the poles (x_1, y_1) and (x_2, y_2).

Solution of the second Problem, 1

The absence of finite singular points with $x \neq x_i$ makes one seek the solution in the form

$$\frac{E(x,y)dx}{(x-x_1)(x_2-x)f_y(x,y)}, \quad E \in \mathbb{C}[x,y],$$

and the absence of points at infinity indicates the fact that the order of the polynomial E cannot exceed $r-1$. Equation

$$f(x_i, y) = 0$$

beside the root $y = y_i$ has $r-1$ more roots; let us denote them as $y_i', \ldots, y_i^{(r-1)}$. If there are no multiple roots among them, then the equations

$$E(x_i, y_i^{(j)}) = 0, \quad i = 1, 2,\ j = 1, \ldots, r-1$$

ensure the absence of singularities at point, different from (x_1, y_1) and (x_2, y_2).

Solution of the second Problem, 2

Calculating the residueses at points (x_1, y_1) and (x_2, y_2) we find two more equations.
As a result, the solution to the second Problem reduces to the solution of a system of linear equations

$$\begin{cases} E(x_i, y_i^{(j)}) = 0, \quad i = 1, 2,\, j = 1, \ldots, r-1 \\ E(x_1, y_1) = (x_2 - x_1) f_y(x_1, y_1), \\ E(x_2, y_2) = (x_2 - x_1) f_y(x_2, y_2). \end{cases}$$

to determinate the coefficients of E.
The coefficients of this system are polynomials with respect $x_i, y_i^{(j)}$ and thus belong to QQbar. The main difference of second Problem from the first one is the necessity to extend the number field.

Algorithm for finding the third kind integrals in the case of simple roots

Input: $f \in \mathbb{Q}[x,y]$, points (x_1,y_1), (x_2,y_2) of the curve f over the field $\overline{\mathbb{Q}}$

Output: differentials udx with p arbitrary coefficients.

1. Calculate the lists R_1 and R_2 of the roots of the equations $f(x_1,y)=0$ and $f(x_2,y)=0$ with respect to y. Delete form they the roots y_1 and y_2.
2. Add symbolic variables c_{ij} and define the expression
$$E = \sum_{i+j\le r} c_{ij}x^i y^j.$$
3. Calculate the lists L of the linear equations from previous slide.
4. Solve the equations L with respect to c_{ij}. Substitute the solution in the expression E and return
$$\frac{E(x,y)dx}{(x-x_1)(x_2-x)f_y(x,y)}.$$

Example, input

Consider an elliptical curve

$$x^3 - y^3 + 2xy + x - 2y + 1 = 0.$$

and construct the differential of the third kind with pols at points with abscissa $x = 0$ and $x = 1$.

```
sage: var("x,y,dx")
sage: f=x^3-y^3+2*x*y+x-2*y+1
sage: x1=0
sage: x2=1
sage: y1=QQbar[y](f.subs(x=x1)).roots(multiplicities=False)
sage: y2=QQbar[y](f.subs(x=x2)).roots(multiplicities=False)
```

Example, first steps

First three steps of Algorithm was realized as function `iii_eqs`, which return six linear equations with six unknowns $c_0, \ldots, c_5$:

```
sage: iii_eqs(f,[x1,y1],[x2,y2])
[c0 + 0.4533976515164038?*c1 + 0.2055694304005904?*c2 +
 2.616708291201771?, ...]
```

To solve system of equations, Sage uses a standard function `solve`, which does not support the operation with algebraic numbers Therefore, we proceeded to matrices over the field of algebraic numbers and tried to solve the system of linear equations by means of function `solve_right`. However, this function did not cope with this system in a reasonable amount of time, returning the warning: `increasing stack size to 32000000`.

Symmetrization of the obtained equations

The system of linear equations consists of two subsystems of the form

$$E(x_i, y_i^{(j)}; c_0, \dots) = b_{i,j}, \quad j = 1, 2, \dots r, \tag{1}$$

where $y_i^{(j)}$ is the set of roots of equation $f(x_i, y) = 0$ with respect to y.

We can obtain from this system r consequences that are symmetric with respect to permutations of the roots of the equation $f(x_i, y) = 0$:

$$\sum_{j=1}^{r} (y_i^{(j)})^k E(x_i, y_i^{(j)}; c_0, \dots) = b_{i,j}, \quad k = 0, 1, \dots r-1. \tag{2}$$

System (2) is equivalent to original system (1), since the new system is obtained from the old one by multiplying by the Vandermonde matrix, the determinant of which in the case of simple roots considered is nonzero.

Example, first steps with symmetrization

After implementing the first three steps and the symmetrization in the form of function `iii_eqs_sym`, we obtain six linear equations with six unknowns $c_0, \dots, c_5$:

```
sage: iii_eqs_sym(f,[x1,y1],[x2,y2])
[3*c0 - (4)*c2 + 2.616708291201771?,
 -(4)*c1 + (3.000000000000000? + 0.?e-36*I)*c2
 + 1.186409393934385?,
 -4*c0 + 3*c1 + (8.000000000000000? + 0.?e-36*I)*c2
 + 0.537915232946850 0?,
 3*c0 + 3*c3 + 3*c5 + 6.240251469155713?,
 9*c2 + 9, 9*c1 + 9*c4 + 12.98024613276667 6?]
```

Example, forced rationalization

Now the drawbacks of the realization of the field of algebraic numbers are obvious: one of coefficients in the second equation is not identified as rational.
When operating with algebraic numbers, there is no rounding error and you can verify its rationality using standard tools:

```
sage: eqs=iii_eqs_sym(f,[x1,y1],[x2,y2])
sage: eqs[1]
-(4)*c1 + (3.000000000000000? + 0.?e-36*I)*c2
+ 1.186409393934385?
sage: eqs[1].coefficient(c2)
3
```

Example, solving of SLAE

The symmetrized SLAE is solved without noticeable expenditure of time.
For convenience we constructed our own user function `lsolve` which reduces the equations to the matrix form and solves them by means of function `solve_right`:

```
sage: lsolve(eqs,[c0,c1,c2,c3,c4,c5])
[c0 == -2.205569430400590?,
c1 == -0.4533976515164038?,
c2 == -1, c3 == 0.1254856073486862?,
c4 == -0.9888519187910046?, c5 == 0]
```

This function, like the function `solve_right` itself, returns a partial solution.

Example, realization of algorithm

The entire Algorithm is implemented as function iii that returns the differential of the third kind, defined up to a linear combination of differentials of the first kind:

```
sage: iii(f,[x1,y1],[x2,y2])
( -0.988851918791004 6?* x * y - y ^2 +
0.1254856073486862?* x - 0.4533976515164038?* y -
2.205569430400590?) * dx /((3* y ^2 - 2* x + 2) *( x - 1) *
```

In the example considered, a visually graspable expression is obtained.

Fundamental function

The theory of algebraic curves is based on a seemingly very simple problem.

Problem

Given an indecomposable polynomial $f \in \mathbb{C}[x, y]$ defining a projective curve C, and s points (x_i, y_i) on this curve. It is required to construct a non-constant rational function $g \in \mathbb{C}(x, y)$, having poles only at given points and only of the first order.

If the points are not chosen in a special way, then this problem is solvable only if the number r is greater than some boundary p, remarkably equal to the genus of the curve.

Definition

The points $(x_1, y_1), \dots (x_p, y_p)$ of the curve C will be called points in general position if the third Problem has no solution.

Fundamental function

Definition

We fix p points (a_i, b_i) in general position on the curve C. The fundamental function is the solution of the third Problem for the set of points obtained by adding one more point to these p points, which will be denoted below as (x', y'). For normalization, it is assumed that the residue at this point is equal to -1.

The definition is convenient for proving the existence of a fundamental function [Weierstrass, ch. 2].

Connection between the fundamental function and Abelian integrals

Let udx be an Abelian differential of the third kind with first-order poles (x_1, y_1) and (x_2, y_2) and residues 1 and -1, and let h be the fundamental function.
Then the product $hudx$ has poles of the first order at $p+3$ points and the residue theorem yields

$$-h|_{(x_1,y_1)} + h|_{(x_2,y_2)} + u|_{(x',y')} + \sum_{i=1}^{p} c_i u|_{(a_i,b_i)} = 0,$$

where c_i is the residue of the fundamental function at the point (a_i, b_i).

Connection between the fundamental function and Abelian integrals, 2

Since the fundamental function is defined up to an additive constant, it is possible to assume that h is zero at the point (x_2, y_2), then

$$h|_{(x_1,y_1)} = u|_{(x',y')} + \sum_{i=1}^{p} c_i u|_{(a_i,b_i)}.$$

Since a differential of the third kind is defined up to p constants, it is always possible to ensure that $u|_{(a_i,b_i)} = 0$. But then

$$h|_{(x_1,y_1)} = u|_{(x',y')},$$

i.e., the fundamental function multiplied by dx', as a function of the point (x', y') is a differential of the third kind.

Calculating the value of the fundametal function at a given point

The formula

$$h|_{(x_1,y_1)} = u|_{(x',y')}$$

allows calculating the value of the fundamental function at almost any point according to presented above Algorithm.
Consider again the elliptic curve

$$x^3 - y^3 + 2xy + x - 2y + 1 = 0$$

of genus 1, at which we take the point $(1, y_2)$ as a zero of the fundamental function, and the point $(2, b_1)$ as an additional pole (a_1, b_1). Find the value of the fundamental function at point $(0, y_1)$.

Example

We define the ordinates of the points so that they fall on the curve

```
sage: f=x^3-y^3+2*x*y+x-2*y+1
sage: x1=0
sage: x2=1
sage: a1=2
sage: xx=3
sage: y1=QQbar[y](f.subs(x=x1)).roots(multiplicities=False)
sage: y2=QQbar[y](f.subs(x=x2)).roots(multiplicities=False)
sage: b1=QQbar[y](f.subs(x=a1)).roots(multiplicities=False)
sage: yy=QQbar[y](f.subs(x=xx)).roots(multiplicities=False)
```

Function `haupt_fuction_eval` returns the value of the fundamental function at point (x_1, y_1):

```
sage: haupt_fuction_eval(f,[x1,y1],[x2,y2],[xx,yy],
[[a1,b1]])
0.0812732274979057?
```

Example, difficulties

```
sage: haupt_fuction_eval(f,[x1,y1],[x2,y2],[xx,yy],
[[a1,b1]])
0.0812732274979057?
```

Unfortunately, a very unexpected difficulty arises here: the value of the fundamental function is an algebraic number, the minimum polynomial for which cannot be calculated in a reasonable time (the warning is `increasing stack size to 64000000`).
In the process of designing the function `haupt_fuction_eval` we faced a number of problems that led to the same warning and offered no possibility to finish the calculations. We managed to avoid them by dividing a number of symbolic expressions into parts, each calculated separately. We had to apply the method `subs` to individual elements rather than to lists.

Conclusion

Summarizing the above, it can be argued that the implication of the algebraic number field in Sage really allows, at least in simple examples, to implement Weierstrass's algorithms almost as described in his Lectures.
Unfortunately, the instrument for working with `QQbar` in Sage did not live up to our hopes. Certainly, instead of `QQbar` we can use a tower of the number fields. The algorithm will retain its appearance, but its implementation will be longer.

The End

Additional materials: `https://malykhmd.neocities.org`