On the computation of Abelian differential of the third kind

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Abstract. We consider the construction of the fundamental function and Abelian differentials of the third kind on a plane algebraic curve over the field of complex numbers that has no singular points. The algorithm for constructing differentials of the third kind is described in Weierstrass's Lectures. The article discusses its implementation in the Sage computer algebra system. The specificity of this algorithm, as well as the very concept of the differential of the third kind, implies the use of not only rational numbers, but also algebraic ones, even when the equation of the curve has integer coefficients. Sage has a built-in algebraic number field tool that allows implementing Weierstrass's algorithm almost verbatim. The simplest example of an elliptic curve shows that it requires too many resources, going far beyond the capabilities of an office computer. Then the symmetrization of the method is proposed and implemented, which solves the problem and allows significant economy of resources. The algorithm for constructing a differential of the third kind is used to find the value of the fundamental function according to the duality principle. Examples explored in the Sage system are provided.

Of all the known approaches to Abelian integrals, Weierstrass's approach was the most constructive. In Ref. [1], we tried to show that the normal form of representation of Abelian integrals proposed in the lectures gives solutions to a number of classical problems and its implementation in computer algebra systems would be very useful. The key problem on this way, both in the 19th century and now, is the construction of the fundamental function (Hauptfuktion) or, which is also due to the duality principle, the differential of the third kind (Art), the construction algorithm of which is described in the last chapter of Part 1 of the Weierstrass Lectures [2], published in 1902 by Hettner and Knoblauch. There are no examples of using the algorithm in the text.

A characteristic feature of Weierstrass' approach is the use of a large number of irrational numbers, the algorithm for determining which is either described in the text, or more or less obvious. The Sage system has a built-in implementation QQbar of the field of algebraic numbers, so in theory the algorithms from the

Lectures can be implemented as written. However, in practice, symbolic expressions containing a ten of numerical coefficients from the field of algebraic numbers QQbar are very difficult to manipulate. We decided to consider this direct implementation of the algorithms and these expressions themselves and evaluate the difficulties that arise.

Let polynomial f define an algebraic curve C of the order r on the projective plane xy over the field $\mathbb{C}$. Let for simplicity this curve have no singular points.

Definition. A differential of the form udx, $u \in \mathbb{C}(x,y)$ having no singular points is called a differential of the first kind. A differential of the form udx, $u \in \mathbb{C}(x,y)$ is called a differential of the third kind, if it has two singular points, namely, poles of the first order (x_1,y_1) and (x_2,y_2) with residues 1 and -1.

Problem 1. Given a polynomial $f \in \mathbb{Q}[x,y]$, find a non-constant rational function $u \in \overline{\mathbb{C}}(x,y)$ such that udx is a differential of the first kind.

The absence of finite singular points makes one seek the solution in the form

$$\frac{E(x,y)dx}{f_y(x,y)}, \quad E \in \mathbb{C}[x,y],$$

and the absence of singular points at infinity indicates the fact that the order of the polynomial E cannot exceed $r-3$. Since no limitations should be imposed on the coefficients of this polynomial, the set of differentials of the first kind has the dimension

$$p = \frac{(r-1)(r-2)}{2},$$

which is called a genus of the curve. For the basis of this space one can take differentials with the coefficients form the field $\mathbb{Q}$, rather than from its algebraic closure. Therefore, when constructing differentials of the first kind it is possible and necessary to work over the field $\mathbb{Q}$.

Algorithms for calculating a basis for the space of differentials of the first kind for planar curves, including those having singular points, have been proposed both in classical books and in present-day papers [3]. At present they are implemented in the systems Maple (AlgCurves, CASA) and Sage.

Problem 2. Given a indecomposable polynomial $f \in \mathbb{Q}[x,y]$, defining a projective curve C, and two points (x_1,y_1) and (x_2,y_2) on this curve, and $x_1,x_2,y_1,y_2 \in \overline{\mathbb{Q}}$. It is required to construct a non-constant rational function $u \in \mathbb{C}(x,y)$ such that udx is a differential of the third kind with the poles (x_1,y_1) and (x_2,y_2).

The addition to the differential of a linear combination of differentials of the first kind does not give rise to new singularities of change of residues, therefore, the solution of Problem 2 is defined to a linear combination of p differentials of the first kind.

The absence of finite singular points with $x \neq x_i$ makes one seek the solution in the form

$$\frac{E(x,y)dx}{(x-x_1)(x_2-x)f_y(x,y)}, \quad E \in \mathbb{C}[x,y],$$

and the absence of points at infinity indicates the fact that the order of the polynomial E cannot exceed $r-1$. Equation

$$f(x_i, y) = 0$$

beside the root $y = y_i$ has $r-1$ more roots; let us denote them as $y_i', \dots, y_i^{(r-1)}$. If there are no multiple roots among them, then the equations

$$E(x_i, y_i^{(j)}) = 0, \quad i = 1, 2,\ j = 1, \dots, r-1$$

ensure the absence of singularities at point, different from (x_1, y_1) and (x_2, y_2). The conditions for residues at these points give two more equations:

$$E(x_1, y_1) = (x_2 - x_1) f_y(x_1, y_1), \quad E(x_2, y_2) = (x_2 - x_1) f_y(x_2, y_2).$$

Thus, the solution to Problem 2 reduces to the solution of a system of linear equations with coefficients from QQbar, and the main difference of Problem 2 from Problem 1 is the necessity to extend the number field.

We wrote a directi realization of the described method in Sage and applied it to an elliptic curve

$$x^3 - y^3 + 2xy + x - 2y + 1 = 0.$$

The solution of Problem 2 led to six linear equations with six unknowns $c_0, \dots, c_5$. To solve systes of equations, Sage uses a standard function `solve`, which does not support the operation with algebraic numbers. Therefore, we proceeded to matrices over the field of algebraic numbers and tried to solve the system of linear equations by means of function `solve_right`. However, this function did not cope with this system in a reasonable amount of time.

Fortunately, the system of equations consists of two subsystems of the form

$$E(x_i, y_i^{(j)}; c_0, \dots) = b_{i,j}, \quad j = 1, 2, \dots r, \tag{1}$$

where $y_i^{(j)}$ is te set of roots of equation $f(x_i, y) = 0$ with respect to y. It can be symmetrized and its solution can be reduced to inverting matrices with rational coefficient. In the example considered, a visually graspable expression is obtained

```
( -0.9888519187910046?* x * y - y ^2 +
0.1254856073486862?* x - 0.4533976515164038?* y -
2.205569430400590?) * dx /((3* y ^2 - 2* x + 2) *( x - 1) * x)
```

Thus, such symmetrization is quite enough for efficient implementation of the method for constructing a differential of the third kind, proposed in Weierstrass's Lectures.

The next step in implementing algorithms, proposed in Weierstrass's Lectures, is the construction of the fundamental function. For this purpose, it is sufficient to construct a differential of the third kind with a movable pole. To execute symmetrization in this case, too, we intend to use a perfect tool — the package Symmetric Functions for Sage, which allows expressing a symmetric function from a ring $K[x_1, \dots, x_n]$ as a linear combination of elementary symmetric functions.

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References

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