Tropical Optimization Techniques for Solving Multicriteria Problems in Decision Making

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Outline

Tropical Optimization

- Tropical optimization focuses on optimization problems that are formulated and solved in the framework of tropical algebra
- Tropical algebra is concerned with the theory and application of semirings and semifields with idempotent operations
- An operation is idempotent, if applied to operands of the same value, it returns this value as output (example: $\max(x, x) = x$)
- Methods and techniques of tropical optimization find application in many areas to offer new solutions to various classical and novel problems
- As an application example one can consider decision-making problems of deriving ratings (priorities) of alternatives from pairwise comparisons

Single-Criterion Pairwise Comparison Problem

- Consider n alternatives of making a decision that are compared in pairs
- This results in a pairwise comparison matrix $\boldsymbol{C} = (c_{ij})$ where the entry $c_{ij} > 0$ shows that alternative i is c_{ij} times more preferable than j
- The matrix $\boldsymbol{C}$ is symmetrically reciprocal, i.e. $c_{ij} = 1/c_{ji}$ for all i, j
- Given a pairwise comparison matrix $\boldsymbol{C}$, the problem of interest is to calculate individual ratings (scores, priorities, weights) of alternatives
- A pairwise comparison matrix $\boldsymbol{C}$ is consistent if $c_{ij} = c_{ik}c_{kj}$ for all i, j, k
- If a pairwise comparison matrix $\boldsymbol{C} = (c_{ij})$ is consistent, then there exists a positive vector $\boldsymbol{x} = (x_i)$ such that $c_{ij} = x_i/x_j$ for all i, j
- Moreover, the vector $\boldsymbol{x}$ (defined up to a positive factor) can be taken as a vector of individual ratings of alternatives and thus solve the problem

Single-Criterion Pairwise Comparison Problem

- The matrices of pairwise comparisons that appear in practice are usually inconsistent, which makes the problem of evaluating ratings nontrivial
- The solution techniques include heuristic methods that do not guarantee the optimality of solution, but offer results acceptable in practice
- There are approximation methods that provide mathematically justified optimal solutions, which however can involve difficult computations
- The log-Chebyshev approximation technique, which minimizes the Chebyshev distance in logarithmic scale, leads to the problem

$$\min_{\boldsymbol{x}>\boldsymbol{0}} \max_{1\leq i,j\leq n} \left| \log c_{ij} - \log \frac{x_i}{x_j} \right|$$

- Observing that the logarithm to a base greater than 1 monotonically increases, one can reduce the problem to that in the form

$$\min_{\boldsymbol{x}>\boldsymbol{0}} \max_{1\leq i,j\leq n} \frac{c_{ij}x_j}{x_i}$$

Single-Criterion Pairwise Comparison Problem

- Suppose that there are constraints imposed on the absolute ratings of alternatives in the form of two-sided bounds on ratios between the ratings
- Given a matrix $\boldsymbol{B} = (b_{ij})$ where $b_{ij} \geq 0$ shows that alternative i must be not less than b_{ij} times better than j, the constraints are written as

$$\max_{1 \leq j \leq n} b_{ij} x_j \leq x_i, \qquad i = 1, \ldots, n$$

- Combining the objective function with the constraints yields the problem

$$\begin{aligned} \min_{\boldsymbol{x} > \boldsymbol{0}} \quad & \max_{1 \leq i,j \leq n} \frac{c_{ij} x_j}{x_i}; \\ \text{s.t.} \quad & \max_{1 \leq j \leq n} b_{ij} x_j \leq x_i, \qquad i = 1, \ldots, n \end{aligned}$$

Multicriteria Pairwise Comparison Problem

- Suppose n alternatives are compared in pairs according to m criteria
- Let $\boldsymbol{C}_l = (c_{ij}^{(l)})$ be a given matrix of pairwise comparisons of alternatives, obtained according to criterion $l = 1, \dots, m$
- Let $\boldsymbol{B} = (b_{ij})$ be a matrix of constraints on the ratings of alternatives
- Consider the problem of finding a vector $\boldsymbol{x} = (x_i)$ of the ratings
- Application of the log-Chebyshev approximation yields the problem

$$\begin{aligned} \min_{\boldsymbol{x} > \boldsymbol{0}} \quad & \left(\max_{1 \leq i,j \leq n} \frac{c_{ij}^{(1)} x_j}{x_i}, \dots, \max_{1 \leq i,j \leq n} \frac{c_{ij}^{(m)} x_j}{x_i} \right); \\ \text{s.t.} \quad & \max_{1 \leq j \leq n} b_{ij} x_j \leq x_i, \qquad i = 1, \dots, n \end{aligned}$$

- Below we consider solutions based on the max-ordering, lexicographic ordering and lexicographic max-ordering principles of optimality
- We offer complete analytical solutions that are obtained in the framework of tropical (idempotent) algebra in a compact vector form

Max-Algebra

- Max-algebra is the set of nonnegative reals equipped with addition $\oplus$ defined as $x \oplus y = \max(x, y)$, and multiplication defined as usual
- Addition has the idempotent property: $x \oplus x = \max(x, x) = x$; multiplication is invertible for all nonzero x and distributes over addition
- The matrix operations follow the standard rules with $+$ replaced by $\oplus$
- The zero vector denoted by $\mathbf{0}$, and identity matrix denoted by $\boldsymbol{I}$ are of the same form as in the conventional algebra
- For any square matrix $\boldsymbol{A} = (a_{ij})$ of order n, the trace is given by

$$\operatorname{tr} \boldsymbol{A} = a_{11} \oplus \cdots \oplus a_{nn}$$

- The power notation indicates repeated multiplication of a matrix by itself
- A tropical analogue of the matrix determinant is defined as

$$\operatorname{Tr}(\boldsymbol{A}) = \operatorname{tr} \boldsymbol{A} \oplus \cdots \oplus \operatorname{tr} \boldsymbol{A}^n$$

- If $\operatorname{Tr}(\boldsymbol{A}) \leq 1$, then the Kleene star operator is calculated as

$$\boldsymbol{A}^* = \boldsymbol{I} \oplus \boldsymbol{A} \oplus \cdots \oplus \boldsymbol{A}^{n-1}$$

Tropical Representation

Conventional algebra representation:	Max-algebra scalar representation:	Max-algebra vector representation:
$\max\limits_{1\le i,j\le n} \dfrac{c_{ij}x_j}{x_i}$	$\bigoplus\limits_{1\le i,j\le n} x_i^{-1}c_{ij}x_j$	$\boldsymbol{x}^{-}\boldsymbol{C}\boldsymbol{x}$
$\max\limits_{1\le j\le n} b_{ij}x_j \le x_i$	$\bigoplus\limits_{1\le j\le n} b_{ij}x_j \le x_i$	$\boldsymbol{B}\boldsymbol{x} \le \boldsymbol{x}$

Representation of the multicriteria optimization problem:

$$\begin{aligned} \min_{\boldsymbol{x}>\boldsymbol{0}} \quad & \left(\max_{1\le i,j\le n} \frac{c_{ij}^{(1)}x_j}{x_i}, \dots, \max_{1\le i,j\le n} \frac{c_{ij}^{(m)}x_j}{x_i}\right); \\ \text{s.t.} \quad & \max_{1\le j\le n} b_{ij}x_j \le x_i, \qquad i = 1,\dots,n \end{aligned}$$

(conventional algebra)

$$\begin{aligned} \min_{\boldsymbol{x}>\boldsymbol{0}} \quad & (\boldsymbol{x}^{-}\boldsymbol{C}_1\boldsymbol{x}, \dots, \boldsymbol{x}^{-}\boldsymbol{C}_m\boldsymbol{x}); \\ \text{s.t.} \quad & \boldsymbol{B}\boldsymbol{x} \le \boldsymbol{x} \end{aligned}$$

(max-algebra)

Max-Ordering Solution

- The max-ordering multicriteria optimization technique aims at minimizing the worst value of the scalar objectives in the vector objective function
- This results in replacing the vector objective function by a function of maximum of the vector components, known as Chebyshev scalarization
- Application of Chebyshev scalarization yields the scalar objective function

$$\max_{1\leq l\leq m}\max_{1\leq i,j\leq n}\frac{c_{ij}^{(l)}x_j}{x_i}=\max_{1\leq i,j\leq n}\frac{c_{ij}x_j}{x_i},\qquad c_{ij}=\max_{1\leq l\leq m}c_{ij}^{(l)}$$

- The max-ordering optimality principle leads to the constrained problem

$$\begin{aligned}\min_{\boldsymbol{x}>\mathbf{0}}\quad &\max_{1\leq i,j\leq n}\frac{c_{ij}x_j}{x_i};\\ \text{s.t.}\quad &\max_{1\leq j\leq n}b_{ij}x_j\leq x_i,\qquad i=1,\dots,n\end{aligned}$$

- The max-ordering solution is known to be week Pareto-optimal for the original multicriteria problem, and becomes Pareto optimal if it is unique

Max-Ordering Solution

Theorem

Let C_l *for all* $l = 1, \ldots, m$ *be matrices such that* $\mathrm{Tr}(C_l) \neq 0$*, and* B *be a matrix such that* $\mathrm{Tr}(B) \leq 1$*. With* $A = C_1 \oplus \cdots \oplus C_m$*, define the scalar*

$$\theta = \bigoplus_{k=1}^{n} \bigoplus_{0 \leq i_1 + \cdots + i_k \leq n-k} \mathrm{tr}^{1/k}(AB^{i_1} \cdots AB^{i_k}).$$

Then, all max-ordering solutions of the problem are given by

$$x = (\theta^{-1} A \oplus B)^{*} u, \qquad u \neq 0$$

Lexicographic Ordering Solution

- Lexicographic optimization considers the objective functions in a hierarchical order based on some ranking of objectives
- Suppose the objectives are numbered in such a way that objective 1 has the highest rank, objective 2 has the second highest and so on
- The lexicographic approach first minimizes function 1 and examine the set of solutions to check whether the solution consists of a unique vector
- If the solution obtained is unique (up to a positive factor), it is taken as the solution of the overall multiobjective problem
- Otherwise function 2 is minimized over all solutions of the first problem, and the procedure continues until a unique solution is obtained

Lexicographic Ordering Solution

- To apply the lexicographic ordering technique, we first take the initial feasible solution set given by the inequality constraints as follows:

$$X_0 = \left\{ \boldsymbol{x} > \boldsymbol{0} \ : \ \max_{1 \leq j \leq n} b_{ij} x_j \leq x_i, \quad i = 1, \ldots, n \right\}$$

- Then, we obtain the solution set X_s for each problem

$$\min_{\boldsymbol{x} \in X_{s-1}} \quad \max_{1 \leq i,j \leq n} \frac{c_{ij}^{(s)} x_j}{x_i}, \qquad s = 1, \ldots, m$$

- The solution procedure stops as soon as the set X_s consists of a single solution vector or all m scalar objective functions are examined
- The last found set X_s is taken as the lexicographic solution

Lexicographic Ordering Solution

Theorem

Let $\boldsymbol{C}_l$ *for all* $l = 1, \ldots, m$ *be matrices such that* $\mathrm{Tr}(\boldsymbol{C}_l) \neq 0$ *, and* $\boldsymbol{B}$ *be a matrix such that* $\mathrm{Tr}(\boldsymbol{B}) \leq 1$ *. With* $\boldsymbol{B}_0 = \boldsymbol{B}$ *, define the recurrence relations*

$$\theta_s = \bigoplus_{k=1}^{n} \bigoplus_{0 \leq i_1 + \cdots + i_k \leq n-k} \mathrm{tr}^{1/k}(\boldsymbol{C}_s \boldsymbol{B}_{s-1}^{i_1} \cdots \boldsymbol{C}_s \boldsymbol{B}_{s-1}^{i_k}),$$

$$\boldsymbol{B}_s = \theta_s^{-1} \boldsymbol{C}_s \oplus \boldsymbol{B}_{s-1}, \qquad s = 1, \ldots, m.$$

Then, all lexicographic ordering solutions of the problem are given by

$$\boldsymbol{x} = \boldsymbol{B}_m^* \boldsymbol{u}, \qquad \boldsymbol{u} \neq \boldsymbol{0}.$$

Note that in practice the derivation of the solution can be stopped due to a unique solution obtained at some step s to avoid redundant computations

Lexicographic Max-Ordering Solution

- This approach combines the lexicographic ordering and max-ordering into one procedure to improve the accuracy of the max-ordering approach
- The technique involves several steps, each finding the max-ordering solution of a problem with less objectives and smaller feasible set
- The first step coincides with the max-ordering solution of the constrained problem with m objectives and the feasible set given by the constraints
- Each step takes the solution set from the previous step as a current feasible solution set and selects objectives that can be further minimized
- A scalar objective function is included in the current function if it has its minimum value over the current feasible set below the minimum obtained

Lexicographic Max-Ordering Solution

- To describe the solution, we use the symbol I_s to denote the set of indices of scalar objective functions involved at step s, and set

$$I_0 = \{1, \ldots, m\}, \qquad X_0 = \left\{ \boldsymbol{x} > \boldsymbol{0} \;:\; \max_{1 \leq j \leq n} b_{ij} x_j \leq x_i, \quad i = 1, \ldots, n \right\}$$

- We derive the minimum θ_s and solution set X_s of the problem

$$\min_{\boldsymbol{x} \in X_{s-1}} \max_{l \in I_{s-1}} \max_{1 \leq i,j \leq n} \frac{c_{ij}^{(l)} x_j}{x_i}, \qquad s = 1, \ldots, m$$

- At each step s, we also define the index set

$$I_s = \left\{ l \in I_{s-1} \;:\; \theta_s > \min_{\boldsymbol{x} \in X_s} \max_{1 \leq i,j \leq n} \frac{c_{ij}^{(l)} x_j}{x_i} \right\}.$$

- The procedure is completed if either the set X_s reduces to a single solution vector or the condition $I_s = \emptyset$ holds

Lexicographic Max-Ordering Solution

Theorem

Let $\boldsymbol{C}_l$ *for all* $l=1,\ldots,m$ *be matrices such that* $\mathrm{Tr}(\boldsymbol{C}_l)\neq 0$*, and* $\boldsymbol{B}$ *be a matrix such that* $\mathrm{Tr}(\boldsymbol{B})\leq 1$*. With* $\boldsymbol{B}_0=\boldsymbol{B}$ *and* $I_0=\{1,\ldots,m\}$*, define the recurrence relations*

$$\theta_s=\bigoplus_{k=1}^{n}\bigoplus_{0\leq i_1+\cdots+i_k\leq n-k}\mathrm{tr}^{1/k}(\boldsymbol{A}_s\boldsymbol{B}_{s-1}^{i_1}\cdots\boldsymbol{A}_s\boldsymbol{B}_{s-1}^{i_k}),\qquad \boldsymbol{A}_s=\bigoplus_{l\in I_{s-1}}\boldsymbol{C}_l,$$

$$I_s=\left\{l\in I_{s-1}\ :\ \theta_s>\bigoplus_{k=1}^{n}\bigoplus_{0\leq i_1+\cdots+i_k\leq n-k}\mathrm{tr}^{1/n}(\boldsymbol{C}_l\boldsymbol{B}_s^{i_1}\cdots\boldsymbol{C}_l\boldsymbol{B}_s^{i_k})\right\},$$

$$\boldsymbol{B}_s=\theta_s^{-1}\boldsymbol{A}_s\oplus\boldsymbol{B}_{s-1},\qquad s=1,\ldots,m.$$

Then, all lexicographic max-ordering solutions of the problem are given by

$$\boldsymbol{x}=\boldsymbol{B}_m^{*}\boldsymbol{u},\qquad \boldsymbol{u}\neq\boldsymbol{0}$$

Conclusions

- We have considered a problem to find ratings of alternatives compared in pairs under several criteria, subject to constraints on the ratings
- The problem was formulated as the log-Chebyshev approximation of the pairwise comparison matrices by a common consistent matrix
- The approximation problem was represented in the framework of tropical algebra as a multiobjective optimization problem
- We have applied methods and results of tropical optimization to handle the optimization problem according to various principles of optimality
- Complete solutions have been obtained in the sense of the max-ordering, lexicographic ordering and lexicographic max-ordering optimality
- The obtained solutions were given in a compact vector form that is ready for formal analysis and efficient computation