Tropical Optimization Techniques for Solving Multicriteria Problems in Decision Making

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Abstract. We consider a decision-making problem to find ratings of alternatives from pairwise comparisons under several criteria, subject to constraints imposed on the ratings. Given matrices of pairwise comparisons, the problem is formulated as the log-Chebyshev approximation of these matrices by a common consistent matrix (a symmetrically reciprocal matrix of unit rank) that minimizes the approximation errors for all matrices simultaneously. We rearrange the approximation problem as a constrained multiobjective optimization problem of finding a vector that determines the approximating matrix. The optimization problem is then represented in the framework of tropical algebra. We apply methods and results of tropical optimization to solve the problem according to various principles of optimality, including the max-ordering, lexicographic ordering and lexicographic max-ordering optimality.

Introduction

Tropical optimization constitutes an important research and application domain of tropical (idempotent) mathematics [1, 2, 3], which focuses on optimization problems that are formulated and solved in the framework of semirings and semifields with idempotent addition. Methods and techniques of tropical optimization find application in many areas, including engineering, computer science and operations research, where they offer new solutions to various classical and novel problems. As an application example one can consider decision-making problems of deriving priorities of alternatives from pairwise comparisons [4, 5, 6].

In this paper, we consider a decision-making problem to find absolute ratings (scores, priorities, weights) of alternatives, which are compared in pairs under several criteria, subject to constraints in the form of two-sided bounds (box-constraints) on ratios between the ratings. Given matrices of pairwise comparisons made according to the criteria, the problem is formulated as the log-Chebyshev approximation of these matrices by a common consistent matrix (a symmetrically reciprocal matrix of unit rank) that minimizes the approximation errors for all matrices simultaneously. We rearrange the approximation problem as a constrained multiobjective optimization problem of finding a vector that determines the approximating consistent matrix.

The optimization problem is then represented in the framework of tropical algebra. We apply methods and results of tropical optimization to handle the multiobjective optimization problem according to various principles of optimality [7, 8]. Complete solutions in the sense of the max-ordering, lexicographic ordering and lexicographic max-ordering optimality are obtained, which are given in a compact vector form ready for formal analysis and efficient computation.

1. Single-Criterion Pairwise Comparison Problem

Suppose that n alternatives are compared in pairs, which results in a pairwise comparison matrix $\boldsymbol{C} = (c_{ij})$ where the entry $c_{ij} > 0$ shows that alternative i is c_{ij} times more preferable than alternative j. The matrix $\boldsymbol{C}$ is assumed to be symmetrically reciprocal, which means that $c_{ij} = 1/c_{ji}$ for all $i, j = 1, \ldots, n$. Given a pairwise comparison matrix $\boldsymbol{C}$, the problem of interest is to calculate individual ratings (scores, priorities, weights) of alternatives.

A pairwise comparison matrix $\boldsymbol{C}$ is referred to as consistent if the condition $c_{ij} = c_{ik}c_{kj}$ holds for all i, j, k. If a pairwise comparison matrix $\boldsymbol{C}$ is consistent, then it is not difficult to verify that there exists a positive vector $\boldsymbol{x} = (x_i)$ whose entries determine the entries of $\boldsymbol{C}$ by the relation $c_{ij} = x_i/x_j$ valid for all i, j. It directly follows from this relation that the vector $\boldsymbol{x}$, which is defined up to a positive factor, can be taken as a vector of absolute ratings of alternatives and thus gives the solution of the pairwise comparison problem.

The matrices of pairwise comparisons that appear in real-world problems are commonly not consistent, which makes the problem of evaluating absolute ratings nontrivial. The solution techniques available to handle the problem include heuristic methods that do not guarantee the optimality of solution, but offer results acceptable in practice, and approximation methods that provide mathematically justified optimal solutions, which however can involve difficult computations.

An approximation technique that minimizes the Chebyshev distance in logarithmic scale (a log-Chebyshev approximation) are proposed in [4]. The method is to find positive vectors $\boldsymbol{x} = (x_i)$ that solve the problem

$$\min_{\boldsymbol{x} > \boldsymbol{0}} \max_{1 \leq i,j \leq n} \left| \log c_{ij} - \log \frac{x_i}{x_j} \right|.$$

Suppose now that there are constraints imposed on the absolute ratings of alternatives in the form of two-sided bounds on ratios between the ratings. Given a matrix $\boldsymbol{B} = (b_{ij})$ where $b_{ij} \geq 0$ shows that alternative i must be considered not less than b_{ij} times better than j, the constraints are given by the inequalities

$$\max_{1\leq j\leq n} b_{ij}x_j \leq x_i, \qquad i=1,\dots,n.$$

Observing that the logarithm to a base greater than 1 monotonically increases, one can rewrite the objective function in the problem as

$$\max_{1\leq i,j\leq n}\left|\log c_{ij} - \log\frac{x_i}{x_j}\right| = \log\max_{1\leq i,j\leq n}\frac{c_{ij}x_j}{x_i}.$$

The logarithmic function on the left-hand side attains its maximum where its argument is maximal, which allows us to remove the logarithm from the objective function to solve the equivalent problem

$$\begin{aligned} &\min_{\boldsymbol{x}>\boldsymbol{0}} && \max_{1\leq i,j\leq n}\frac{c_{ij}x_j}{x_i}; \\ &\text{s.t.} && \max_{1\leq j\leq n} b_{ij}x_j \leq x_i, \qquad i=1,\dots,n. \end{aligned}$$

2. Multicriteria Pairwise Comparison Problems

Assume that n alternatives are compared in pairs according to m criteria. For each criterion $l=1,\dots,m$, the results of pairwise comparisons are given by a matrix $\boldsymbol{C}_l=(c_{ij}^{(l)})$ of order n. The problem is to find a vector $\boldsymbol{x}=(x_i)$ of ratings subject to constraints given by a matrix $\boldsymbol{B}=(b_{ij})$ of order n. Application of the log-Chebyshev approximation technique yields the problem

$$\begin{aligned} &\min_{\boldsymbol{x}>\boldsymbol{0}} && \left(\max_{1\leq i,j\leq n}\frac{c_{ij}^{(1)}x_j}{x_i},\dots,\max_{1\leq i,j\leq n}\frac{c_{ij}^{(m)}x_j}{x_i}\right); \\ &\text{s.t.} && \max_{1\leq j\leq n} b_{ij}x_j \leq x_i, \qquad i=1,\dots,n. \end{aligned} \tag{1}$$

In the rest of this section, we consider three common approaches to handle problem (1), which result in different procedures to find the solution set X. The solution techniques used are based on the max-ordering, lexicographic ordering and lexicographic max-ordering principles of optimality [7].

2.1. Max-Ordering Solution

Max-ordering optimization aims at minimizing the worst value of the objective functions, and leads to replacing the vector of objective functions by a scalar function given by the maximum of the objective functions (Chebyshev scalarization).

To solve the constrained problem at (1), we define the feasible solution set

$$X_0 = \left\{\boldsymbol{x}>\boldsymbol{0} \;:\; \max_{1\leq j\leq n} b_{ij}x_j \leq x_i, \quad i=1,\dots,n\right\}.$$

We apply the Chebyshev scalarization to form the objective function

$$\max_{1\leq l\leq m}\max_{1\leq i,j\leq n}\frac{c_{ij}^{(l)}x_j}{x_i} = \max_{1\leq i,j\leq n}\frac{c_{ij}x_j}{x_i}, \qquad c_{ij}=\max_{1\leq l\leq m} c_{ij}^{(l)}.$$

Then, the problem reduces to the constrained minimization problem

$$\min_{\boldsymbol{x}\in X_0} \quad \max_{1\leq i,j\leq n} \frac{c_{ij}x_j}{x_i},$$

which is to solve to obtain the max-ordering solution as the set

$$X_1 = \arg\min_{\boldsymbol{x}\in \boldsymbol{X}_0} \max_{1\leq i,j\leq n} \frac{c_{ij}x_j}{x_i}.$$

Note that the solution obtained by the max-ordering optimization is known to be week Pareto-optimal, and becomes Pareto optimal if it is unique [8].

2.2. Lexicographic Ordering Solution

Lexicographic optimization considers the objective functions in a hierarchical order based on some ranking of objectives. Suppose the objectives are numbered in such a way that objective 1 has the highest rank, objective 2 has the second highest and so on. The lexicographic approach first minimizes function 1 and examine the set of solutions obtained. If the solution obtained is unique (up to a positive factor), it is taken as the solution of the overall multiobjective problem. Otherwise function 2 is minimized over all solutions of the first problem, and the procedure continues until a unique solution is obtained or the problem with function m is solved.

To apply this approach, we first take the initial feasible solution set X_0 defined above, and then obtain the solution set X_s for each problem

$$\min_{\boldsymbol{x}\in X_{s-1}} \quad \max_{1\leq i,j\leq n} \frac{c_{ij}^{(s)}x_j}{x_i}, \qquad s = 1,\ldots,m.$$

The solution procedure stops as soon as the set X_s consists of a single solution vector or all m scalar objective functions are examined. The last found set X_s is taken as the lexicographic solution for the problem.

2.3. Lexicographic Max-Ordering Solution

This approach combines the lexicographic ordering and max-ordering into one procedure that improves the accuracy of the assessment provided by the max-ordering approach. The procedure consists of several steps, each of which finds the max-ordering solution of a reduced problem that has a lower multiplicity of objectives and smaller feasible set. The first solution step coincides with the above described max-ordering solution of the constrained problem with m objectives and the feasible solution set given by the constraints. Each subsequent step takes the solution from the previous step as a current feasible solution set and selects objectives that can be further minimized over the current feasible set, to incorporate into the current vector objective function. A scalar objective function is included in the current function if it has its minimum value over the current feasible set below the minimum of the objective function at the previous step.

To describe the solution, we use the symbol I_s to denote the set of indices of scalar objective functions involved at step s. We initially set $I_0 = \{1,\ldots,m\}$ and define X_0 as above. At each step s, we need to solve the problem

$$\min_{\boldsymbol{x}\in X_{s-1}} \max_{l\in I_{s-1}} \max_{1\leq i,j\leq n} \frac{c_{ij}^{(l)} x_j}{x_i}, \qquad s = 1,\dots,m.$$

where X_{s-1} denotes the solution set of the problem at step $s-1$.

With the minimum value of the objective function at step s denoted by θ_s, we define the index set as follows:

$$I_s = \left\{ l \in I_{s-1} : \ \theta_s > \min_{\boldsymbol{x}\in X_s} \max_{1\leq i,j\leq n} \frac{c_{ij}^{(l)} x_j}{x_i} \right\}.$$

The procedure is completed if either the set X_s reduces to a single solution vector, the condition $I_s = \emptyset$ holds or all m objective functions are examined.

Below, we show how the solutions offered by the above methods can be represented in explicit analytical form using methods and result of tropical mathematics.

3. Preliminary Algebraic Definitions and Notation

Consider a tropical (idempotent) semifield that is defined as the set of nonnegative reals equipped with addition $\oplus$ given by the maximum as $x \oplus y = \max(x,y)$, and multiplication denoted and defined as usual. Addition is idempotent since $x \oplus x = \max(x,x) = x$, and has 0 as the neutral element. Multiplication has 1 as the neutral element, is invertible for all nonzero x and distributes over addition. This tropical semifield is commonly referenced to as the max-algebra.

Matrices and vectors over the max-algebra are routinely introduced. Matrix and vector operations follow the standard entrywise rules with the scalar addition $+$ replaced by $\oplus$. The conjugate of a column vector $\boldsymbol{x} = (x_j)$ is the row vector $\boldsymbol{x}^- = (x_j^-)$ where $x_j^- = x_j^{-1}$ if $x_j \neq 0$, and $x_j^- = 0$ otherwise. The zero vector is denoted by $\mathbf{0}$, and identity matrix by $\boldsymbol{I}$. For any square matrix, the power notation indicates repeated (tropical) multiplication of the matrix by itself.

For any square matrix $\boldsymbol{A} = (a_{ij})$ of order n, the trace is given by

$$\operatorname{tr} \boldsymbol{A} = a_{11} \oplus \cdots \oplus a_{nn}.$$

A tropical analogue of the matrix determinant is defined as

$$\operatorname{Tr}(\boldsymbol{A}) = \operatorname{tr} \boldsymbol{A} \oplus \cdots \oplus \operatorname{tr} \boldsymbol{A}^n.$$

If $\operatorname{Tr}(\boldsymbol{A}) \leq 1$, then the Kleene star operator is calculated as

$$\boldsymbol{A}^* = \boldsymbol{I} \oplus \boldsymbol{A} \oplus \cdots \oplus \boldsymbol{A}^{n-1}.$$

4. Solution of Multicriteria Pairwise Comparison Problems

Consider the multiobjective optimization problem at (1). After rewriting the objective functions and inequality constraints in terms of max-algebra, the problem can be formulated in vector form as follows. Given $(n \times n)$-matrices $\boldsymbol{C}_l$ of pairwise comparisons of n alternatives for criteria $l = 1, \ldots, m$, and nonnegative $(n \times n)$-matrix $\boldsymbol{B}$ of constraints, find positive n-vectors $\boldsymbol{x}$ of ratings that solve the problem

$$\begin{aligned} \min_{\boldsymbol{x} > \boldsymbol{0}} \quad & (\boldsymbol{x}^{-}\boldsymbol{C}_1\boldsymbol{x}, \ldots, \boldsymbol{x}^{-}\boldsymbol{C}_m\boldsymbol{x}); \\ \text{s.t.} \quad & \boldsymbol{B}\boldsymbol{x} \leq \boldsymbol{x}. \end{aligned} \tag{2}$$

Below, we offer max-ordering, lexicographic and lexicographic max-ordering optimal solutions to this problem.

4.1. Max-Ordering Solution

We start with a solution obtained according to the max-ordering optimality.

Theorem 1. *Let $\boldsymbol{C}_l$ for all $l = 1, \ldots, m$ be matrices such that $\mathrm{Tr}(\boldsymbol{C}_l) \neq 0$, and $\boldsymbol{B}$ be a matrix such that $\mathrm{Tr}(\boldsymbol{B}) \leq 1$. With $\boldsymbol{A} = \boldsymbol{C}_1 \oplus \cdots \oplus \boldsymbol{C}_m$, define the scalar*

$$\theta = \bigoplus_{k=1}^{n} \bigoplus_{0 \leq i_1 + \cdots + i_k \leq n-k} \mathrm{tr}^{1/k}(\boldsymbol{A}\boldsymbol{B}^{i_1} \cdots \boldsymbol{A}\boldsymbol{B}^{i_k})$$

and matrix

$$\boldsymbol{G} = (\theta^{-1}\boldsymbol{A} \oplus \boldsymbol{B})^{*}.$$

Then, all max-ordering solutions of problem (2) are given in parametric form by

$$\boldsymbol{x} = \boldsymbol{G}\boldsymbol{u}, \qquad \boldsymbol{u} \neq \boldsymbol{0}.$$

4.2. Lexicographic Ordering Solution

The lexicographic ordering technique solves problem (2) in m steps each minimizing a scalar objective function over a feasible set given by the previous step.

Theorem 2. *Let $\boldsymbol{C}_l$ for all $l = 1, \ldots, m$ be matrices such that $\mathrm{Tr}(\boldsymbol{C}_l) \neq 0$, and $\boldsymbol{B}$ be a matrix such that $\mathrm{Tr}(\boldsymbol{B}) \leq 1$. With $\boldsymbol{B}_0 = \boldsymbol{B}$, define the recurrence relations*

$$\begin{aligned} \theta_s &= \bigoplus_{k=1}^{n} \bigoplus_{0 \leq i_1 + \cdots + i_k \leq n-k} \mathrm{tr}^{1/k}(\boldsymbol{C}_s\boldsymbol{B}_{s-1}^{i_1} \cdots \boldsymbol{C}_s\boldsymbol{B}_{s-1}^{i_k}), \\ \boldsymbol{B}_s &= \theta_s^{-1}\boldsymbol{C}_s \oplus \boldsymbol{B}_{s-1}, \qquad s = 1, \ldots, m; \end{aligned}$$

and the matrix

$$\boldsymbol{G} = \boldsymbol{B}_m^{*}.$$

Then, all max-ordering solutions of problem (2) are given by

$$\boldsymbol{x} = \boldsymbol{G}\boldsymbol{u}, \qquad \boldsymbol{u} \neq \boldsymbol{0}.$$

4.3. Lexicographic Max-Ordering Solution

Similar to the lexicographic ordering solution, we handle problem (2) by solving a series of problems, where each problem has a scalar objective function and inequality constraint provided by the solution of the previous problems. The solution obtained in the framework of max-algebra is described as follows.

Theorem 3. *Let $\boldsymbol{C}_l$ for all $l = 1, \dots, m$ be matrices such that* $\mathrm{Tr}(\boldsymbol{C}_l) \neq 0$*, and* $\boldsymbol{B}$ *be a matrix such that* $\mathrm{Tr}(\boldsymbol{B}) \leq 1$*. With* $\boldsymbol{B}_0 = \boldsymbol{B}$ *and* $I_0 = \{1, \dots, m\}$*, define the recurrence relations*

$$\theta_s = \bigoplus_{k=1}^{n} \bigoplus_{0 \leq i_1 + \cdots + i_k \leq n-k} \mathrm{tr}^{1/k}(\boldsymbol{A}_s \boldsymbol{B}_{s-1}^{i_1} \cdots \boldsymbol{A}_s \boldsymbol{B}_{s-1}^{i_k}), \qquad \boldsymbol{A}_s = \bigoplus_{l \in I_{s-1}} \boldsymbol{C}_l,$$

$$I_s = \left\{ l \in I_{s-1} \ : \ \theta_s > \bigoplus_{k=1}^{n} \bigoplus_{0 \leq i_1 + \cdots + i_k \leq n-k} \mathrm{tr}^{1/n}(\boldsymbol{C}_l \boldsymbol{B}_s^{i_1} \cdots \boldsymbol{C}_l \boldsymbol{B}_s^{i_k}) \right\},$$

$$\boldsymbol{B}_s = \theta_s^{-1} \boldsymbol{A}_s \oplus \boldsymbol{B}_{s-1}, \qquad s = 1, \dots, m;$$

and the matrix

$$\boldsymbol{G} = \boldsymbol{B}_m^{*}.$$

Then, all lexicographic max-ordering solutions of problem (2) *are given by*

$$\boldsymbol{x} = \boldsymbol{G}\boldsymbol{u}, \qquad \boldsymbol{u} \neq \boldsymbol{0}.$$

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