# Algebraic solution of a scheduling problem in project management

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# Introduction

1. One of the main problems of the project management is the problem of drawing up an optimal schedule of jobs in a project.
2. An effective approach to solve such problems is to use methods of the tropical mathematics.
3. Tropical mathematics studies various aspects of theory and application of algebraic systems with idempotent operations.
4. Tropical mathematics is studied, for example, in papers by N.N. Vorobiev, I.V. Romanovsky, V.P. Maslov.
5. In this paper we consider the problem of minimizing the maximum deviation of the start times of jobs from the due dates under given various temporal constraints.

# Basic definitions

Let us consider a project which consists of $n$ jobs performed in parallel. For each job $i = 1, \ldots, n$ we introduce the notation:

1. we denote the start time by $x_i$ and the finish time by $y_i$;
2. $g_i$ and $h_i$ define the earliest and latest allowed start time, as well as $f_i$ defines the latest finish time;
3. $b_{ij}$ denotes the minimum allowed time interval between the start of job $i$ and the start of job $j$;
4. we define the minimum allowed interval between the start time of job $i$ and the finish time of job $j$ by $c_{ij}$;
5. we denote the minimum allowed interval between the finish time of job $i$ and the start time of job $j$ by $d_{ij}$;
6. if the value $b_{ij}$, $c_{ij}$ or $d_{ij}$ is not set, then we will consider it equal to $-\infty$.

# Constraints on jobs performance order

1. Minimum allowed interval between the start time of any two jobs is given by constraints of the type "start-start", and the minimum allowed interval between the finish of job $i$ and the start of job $j$ by constraints "finish-start" in the form of inequalities

$$\max_{1 \le j \le n} (b_{ij} + x_j) \le x_i, \quad \max_{1 \le j \le n} (d_{ij} + y_j) \le x_i \quad i = 1, \dots, n.$$

2. The minimum allowed interval between the start of job $i$ and the end of job $j$ defines constraints of type "start-finish" in the form of equality

$$\max_{1 \le j \le n} (c_{ij} + x_j) = y_i, \quad i = 1, \dots, n.$$

3. Let us write down the constraints on the start and end times of job using the inequalities

$$g_i \le x_i \le h_i, \quad y_i \le f_i, \quad i = 1, \dots, n.$$

# Optimality criteria and the sheduling problem

1. Suppose that for each job $i$, a due date $p_i$ is given, which determines the most desirable start time.
2. It is required to minimize the maximum deviation of start time of jobs from the due dates
3. The optimality criteria for the plan, which need to be mimimized, is written as

$$\max\left(\max_{1\leq i\leq n}(p_i - x_i), \max_{1\leq i\leq n}(x_i - p_i)\right).$$

4. Optimal planning problem under given constraints can be written as

$$\min_{x_i, y_i} \quad \max\left(\max_{1\leq i\leq n}(p_i - x_i), \max_{1\leq i\leq n}(x_i - p_i)\right);$$

$$\max_{1\leq j\leq n}(b_{ij} + x_j) \leq x_i, \quad \max_{1\leq j\leq n}(c_{ij} + x_j) = y_i,$$

$$\max_{1\leq j\leq n}(d_{ij} + y_j) \leq x_i, \quad g_i \leq x_i \leq h_i, \quad y_i \leq f_i, \quad i = 1, \ldots, n.$$

# Idempotent semifield

1. $\langle \mathbb{X}, \mathbb{0}, \mathbb{1}, \oplus, \otimes \rangle$ - idempotent semifield, that is
   - commutative semiring with zero $\mathbb{0}$ and unit $\mathbb{1}$,
   - for any $x \in \mathbb{X}$ the equality $x \oplus x = x$ holds (idempotency).
   - for any $x \neq \mathbb{0}$ there exists $x^{-1}$ such that $x \otimes x^{-1} = \mathbb{1}$ (existence of inverse element).
2. A partial order is defined: $x \leq y$ if and only if $x \oplus y = y$.
3. For any $x \neq \mathbb{0}$ and integer $p > 0$, the power is defined in the usual way: $x^0 = \mathbb{1}$, $x^p = x^{p-1}x$, $x^{-p} = (x^{-1})^p$, $\mathbb{0}^p = \mathbb{0}$. It is assumed that the powers with rational exponents are also defined.
4. An example of an idempotent semifield is the $(\max, +)$-algebra

$$\mathbb{R}_{\max,+} = \langle \mathbb{R} \cup \{-\infty\}, -\infty, 0, \max, + \rangle.$$

# Idempotent matrix algebra

- $\mathbb{X}^{m\times n}$ — the set of matrices which consist of $m$ rows and $n$ columns.
- Matrix and vector operations are performed according to the usual rules with the operations $+$ and $\times$ replaced by $\oplus$ and $\otimes$.
- The matrix $\boldsymbol{I}$ with elements equal to $\mathbb{1}$ on the main diagonal and $\mathbb{0}$ outside is the identity matrix.
- For any square matrix $\boldsymbol{A} = (a_{ij})$ and integer $p > 0$ the power is given by: $\boldsymbol{A}^0 = \boldsymbol{I}$, $\boldsymbol{A}^p = \boldsymbol{A}^{p-1}\boldsymbol{A}$.
- For the matrix $\boldsymbol{A} = (a_{ij}) \in \mathbb{X}^{n\times n}$, we define the functions

$$\operatorname{tr}\boldsymbol{A} = a_{11} \oplus \cdots \oplus a_{nn}, \quad \operatorname{Tr}(\boldsymbol{A}) = \operatorname{tr}\boldsymbol{A} \oplus \cdots \oplus \operatorname{tr}\boldsymbol{A}^n.$$

- Kleene matrix is defined in the form

$$\boldsymbol{A}^* = \boldsymbol{I} \oplus \boldsymbol{A} \oplus \cdots \oplus \boldsymbol{A}^{n-1}.$$

# Idempotent vector algebra

- $\mathbb{X}^n$ — set of column vectors of $n$ elements.
- For any nonzero vector $\boldsymbol{x} \in \mathbb{X}^n$, the transposed vector is denoted as $\boldsymbol{x}^T$.
- A vector without zero elements is called regular.
- The multiplicatively conjugate vector for $\boldsymbol{x} = (x_i)$ is the row vector $\boldsymbol{x}^- = (x_i^-)$, where $x_i^- = x_i^{-1}$ if $x_i \neq \mathbb{0}$ and $x_i^- = \mathbb{0}$ otherwise.
- The partial order is generalized to matrices and vectors and is understood component by component.

# Minimizing of the maximum deviation from the due dates

Consider the problem of minimizing the maximum deviation of start times from the due dates for jobs in terms of the semifield $\mathbb{R}_{\max,+}$. We introduce the following matrices and vectors:

$$\boldsymbol{B} = (b_{ij}), \quad \boldsymbol{C} = (c_{ij}), \quad \boldsymbol{D} = (d_{ij}),$$
$$\boldsymbol{x} = (x_i), \quad \boldsymbol{y} = (y_i), \quad \boldsymbol{f} = (f_i), \quad \boldsymbol{g} = (g_i), \quad \boldsymbol{h} = (h_i), \quad \boldsymbol{p} = (p_i).$$

By replacing arithmetic operations with operations of the semifield $\mathbb{R}_{\max,+}$, we represent the problem in vector form as

$$\begin{aligned} \min_{\boldsymbol{x},\boldsymbol{y}} \quad & \boldsymbol{x}^{-}\boldsymbol{p} \oplus \boldsymbol{p}^{-}\boldsymbol{x}, \\ & \boldsymbol{B}\boldsymbol{x} \leq \boldsymbol{x}, \quad \boldsymbol{C}\boldsymbol{x} = \boldsymbol{y}, \\ & \boldsymbol{D}\boldsymbol{y} \leq \boldsymbol{x}, \quad \boldsymbol{g} \leq \boldsymbol{x} \leq \boldsymbol{h}, \quad \boldsymbol{y} \leq \boldsymbol{f}. \end{aligned}$$

# Minimizing of the maximum deviation from the due dates

The solution of the planning problem is given by the following result.

**Lemma**

Let $\boldsymbol{B}$ and $\boldsymbol{D}$ be matrices, $\boldsymbol{C}$ be a column-regular matrix such that the matrix $\boldsymbol{R} = \boldsymbol{B} \oplus \boldsymbol{D}\boldsymbol{C}$ satisfies $\mathrm{Tr}(\boldsymbol{R}) \leq \mathbb{1}$. Let $\boldsymbol{g}$ be a vector, and $\boldsymbol{f}$ and $\boldsymbol{h}$ be regular vectors such that the vector $\boldsymbol{s}^T = \boldsymbol{f}^- \boldsymbol{C} \oplus \boldsymbol{h}^-$ satisfies the condition $\boldsymbol{s}^T \boldsymbol{R}^* \boldsymbol{g} \leq \mathbb{1}$. Then the minimum value of the objective function in the problem is equal to

$$\theta = (\boldsymbol{p}^- \boldsymbol{R}^* \boldsymbol{p})^{1/2} \oplus \boldsymbol{s}^T \boldsymbol{R}^* \boldsymbol{p} \oplus \boldsymbol{p}^- \boldsymbol{R}^* \boldsymbol{g},$$

and all regular solutions have the form

$$\boldsymbol{x} = \boldsymbol{R}^* \boldsymbol{u}, \qquad \boldsymbol{y} = \boldsymbol{C}\boldsymbol{R}^* \boldsymbol{u},$$
$$\boldsymbol{g} \oplus \theta^{-1} \boldsymbol{p} \leq \boldsymbol{u} \leq ((\boldsymbol{s}^T \oplus \theta^{-1} \boldsymbol{p}^-) \boldsymbol{R}^*)^-.$$

# Numerical example

Let us consider an example of solving the problem of minimizing the maximum deviation from the due dates. Let the project consist of $n = 3$ jobs with the constraints introduced above, given with the notation $\mathbb{0} = -\infty$, by following matrices and vectors

$$\boldsymbol{B} = \begin{pmatrix} 0 & -1 & 1 \\ 0 & -1 & 2 \\ -2 & -3 & 0 \end{pmatrix}, \boldsymbol{C} = \begin{pmatrix} 4 & 2 & 3 \\ 3 & 1 & 2 \\ 2 & 1 & 3 \end{pmatrix}, \boldsymbol{D} = \begin{pmatrix} -4 & -5 & -6 \\ \mathbb{0} & -4 & -7 \\ -5 & \mathbb{0} & -4 \end{pmatrix},$$

$$\boldsymbol{g} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \quad \boldsymbol{h} = \begin{pmatrix} 4 \\ 3 \\ 2 \end{pmatrix} \quad \boldsymbol{f} = \begin{pmatrix} 6 \\ 6 \\ 6 \end{pmatrix}, \quad \boldsymbol{p} = \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}.$$

Let us find auxiliary matrices

$$\boldsymbol{DC} = \begin{pmatrix} 0 & -2 & -1 \\ -1 & -3 & -2 \\ -1 & -3 & -1 \end{pmatrix}, \quad \boldsymbol{R} = \boldsymbol{B} \oplus \boldsymbol{DC} = \begin{pmatrix} 0 & -1 & 1 \\ 0 & -1 & 2 \\ -1 & -3 & 0 \end{pmatrix}.$$

# Numerical example

Then we will calculate the powers of the matrix $\boldsymbol{B} \oplus \boldsymbol{D}\boldsymbol{C}$:

$$\boldsymbol{R}^2 = \begin{pmatrix} 0 & -1 & 1 \\ 1 & -1 & 2 \\ -1 & -2 & 0 \end{pmatrix}, \quad \boldsymbol{R}^3 = \begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & 2 \\ -1 & -2 & 0 \end{pmatrix},$$

and check that $\mathrm{Tr}(\boldsymbol{R}) = \operatorname{tr} \boldsymbol{R} \oplus \operatorname{tr} \boldsymbol{R}^2 \oplus \operatorname{tr} \boldsymbol{R}^3 = 0 \leq \mathbb{1}$.
Let us calculate

$$\boldsymbol{R}^* = \boldsymbol{I} \oplus \boldsymbol{R} \oplus \boldsymbol{R}^2 = \begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & 2 \\ -1 & -2 & 0 \end{pmatrix}, \qquad \boldsymbol{R}^*\boldsymbol{g} = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix},$$

$$\boldsymbol{f}^-\boldsymbol{C} = \begin{pmatrix} -2 & -4 & -3 \end{pmatrix}, \qquad \boldsymbol{f}^-\boldsymbol{C}\boldsymbol{B}^*\boldsymbol{g} = -1, \qquad \boldsymbol{h}^-\boldsymbol{B}^*\boldsymbol{g} = -1,$$

and verify that the condition
$\boldsymbol{s}^T\boldsymbol{R}^*\boldsymbol{g} = (\boldsymbol{f}^-\boldsymbol{C} \oplus \boldsymbol{h}^-)\boldsymbol{R}^*\boldsymbol{g} = -1 < \mathbb{1}$ is satisfied.

# Numerical example

Let us calculate the values $\boldsymbol{p}^{-}\boldsymbol{R}^{*}\boldsymbol{p} = 1$, $\boldsymbol{s}^{T}\boldsymbol{R}^{*}\boldsymbol{p} = 1$, $\boldsymbol{p}^{-}\boldsymbol{R}^{*}\boldsymbol{g} = 0$.
The minimum $\theta = (\boldsymbol{p}^{-}\boldsymbol{R}^{*}\boldsymbol{p})^{1/2} \oplus \boldsymbol{s}^{T}\boldsymbol{R}^{*}\boldsymbol{p} \oplus \boldsymbol{p}^{-}\boldsymbol{R}^{*}\boldsymbol{g}$ is equal to 1.
To write down all solutions $\boldsymbol{x}$ of the problem in parametric form, we find the following matrix and vectors:

$$\boldsymbol{C}\boldsymbol{R}^{*} = \begin{pmatrix} 4 & 3 & 5 \\ 4 & 2 & 4 \\ 2 & 1 & 3 \end{pmatrix}, \quad \boldsymbol{s}^{T} \oplus \theta^{-1}\boldsymbol{p}^{-} = \begin{pmatrix} -2 & -3 & -2 \end{pmatrix},$$

$$\boldsymbol{g} \oplus \theta^{-1}\boldsymbol{p} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \quad ((\boldsymbol{s}^{T} \oplus \theta^{-1}\boldsymbol{p}^{-})\boldsymbol{R}^{*})^{-} = \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}.$$

The minimum in the problem is equal to 1 and reached at
$\boldsymbol{x} = \boldsymbol{R}^{*}\boldsymbol{u}, \quad \boldsymbol{y} = \boldsymbol{C}\boldsymbol{R}^{*}\boldsymbol{u}, \quad \boldsymbol{g} \oplus \theta^{-1}\boldsymbol{p} \leq \boldsymbol{u} \leq ((\boldsymbol{s}^{T} \oplus \theta^{-1}\boldsymbol{p}^{-})\boldsymbol{R}^{*})^{-}$.

# Conclusion

- The problem of minimizing the maximum deviation of the start time of jobs from the due dates was studied.
- The problem has strong constraints of the form "start-start", "start-finish", "finish-start" and boundaries for start and finish times.
- An analytical solution in a compact vector form has been proposed based on methods and results of tropical optimization.