Algebraic solution of a scheduling problem in project management

Gubanov S. A., Design bureau "Luch", software engineer,
segubanov@mail.ru

Abstract. A project scheduling problem is examined, where the maximum deviation of start time of jobs is minimized under various constraints imposed on the start and finish time of jobs. We represent the problem in terms of tropical algebra as a tropical optimization problem, and then obtain a direct solution given in compact vector form.

1. Introduction

One of the main problems of the project management is the problem of drawing up an optimal schedule of jobs in a project [1, 2]. To solve scheduling problems, models and methods of tropical mathematics are used, which studies semirings and semifields with idempotent addition [3, 4, 5]. Scheduling problems are reduced to optimization problems formulated and solved in terms of tropical mathematics (tropical optimization problems) [6, 7, 8, 9]. In this paper we consider the problem of minimizing the maximum deviation of the start times of jobs from the due dates under given various temporal constraints. The problem is represented as a tropical optimization problem, and then solved using a result of the paper [10].

2. Optimal scheduling problem

We consider a problem which arises in project management where an optimal schedule for a project is developed to minimize the maximum deviation of start times of jobs from given due dates.

Let us consider a project which consists of n jobs performed in parallel, subject to time constraints in the form of “start-start”, “start-finish”, and “finish-start” precedence relationships, as well as boundaries for the start and finish times of jobs.

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For each job $i = 1, \dots, n$, we denote the start time by x_i and the finish time by y_i. Let the values g_i and h_i define the earliest and latest allowed start time, as well as f_i defines the latest finish time. These values set boundaries for the start and finish times in the form of the inequalities

$$g_i \leq x_i \leq h_i, \qquad y_i \leq f_i.$$

The "start-start" constraints for the job i are defined in the form of inequalities $b_{ij} + x_j \leq x_i$ for all $j = 1, \dots, n$, where b_{ij} denotes the minimum allowed time interval between the start of job i and the start of j. We put $b_{ij} = -\infty$ if the value b_{ij} is not set. Combining the inequalities over all j gives the equivalent inequality

$$\max_{1 \leq j \leq n} (b_{ij} + x_j) \leq x_i.$$

Let us denote the minimum allowed interval between the start time of i and the finish time of j by c_{ij} ($c_{ij} = -\infty$ if the interval is not specified) and write the "start-finish" constraint in the form of the inequality $c_{ij} + x_j \leq y_i$. We assume that the job finishes immediately when the "start-finish" constraints are satisfied, and then the equality $c_{ij} + x_j = y_i$ is satisfied for at least one j. After combining the inequalities for all j, we obtain

$$\max_{1 \leq j \leq n} (c_{ij} + x_j) = y_i.$$

We denote the minimum allowed interval between the finish time of job i and the start time of j by d_{ij} ($d_{ij} = -\infty$ if no interval is specified). The "finish-start" constraints are written as the inequalities $d_{ij} + y_j \leq x_i$, which are combined into the inequality

$$\max_{1 \leq j \leq n} (d_{ij} + y_j) \leq x_i.$$

Suppose that for each job i, a due date p_i is given, which determines the most desirable start time. We formulate a problem of minimizing the maximum deviation of start time of jobs from the due dates under given constraints as the problem of determining for all $i = 1, \dots, n$ the values x_i and y_i to find

$$\begin{aligned} \min_{x_i, y_i} \quad & \max\left(\max_{1 \leq i \leq n} (p_i - x_i), \max_{1 \leq i \leq n} (x_i - p_i) \right); \\ & \max_{1 \leq j \leq n} (b_{ij} + x_j) \leq x_i, \quad \max_{1 \leq j \leq n} (c_{ij} + x_j) = y_i, \\ & \max_{1 \leq j \leq n} (d_{ij} + y_j) \leq x_i, \quad g_i \leq x_i \leq h_i, \\ & y_i \leq f_i, \quad i = 1, \dots, n. \end{aligned} \tag{1}$$

Below, this problem is formulated in terms of tropical mathematics and solved by using methods of tropical optimization.

3. Elements of tropical mathematics

Let us outline the main definitions and results of tropical (idempotent) mathematics [7, 8, 4, 5], which are used in the next section for describing and solving tropical optimization problems.

Let $\mathbb{X}$ be a set that is closed under the associative and commutative operations of addition $\oplus$ and multiplication $\otimes$, and contain their neutral elements zero $\mathbb{0}$ and one $\mathbb{1}$. Addition is idempotent (for each $x \in \mathbb{X}$ the equality $x \oplus x = x$ holds), while multiplication is distributive with respect to addition and invertible (for any $x \neq \mathbb{0}$ there exists x^{-1} such that $x \otimes x^{-1} = \mathbb{1}$). The algebraic system $\langle \mathbb{X}, \mathbb{0}, \mathbb{1}, \oplus, \otimes \rangle$ is called an idempotent semifield. The sign $\otimes$ of the multiplication operation will be omitted from now on.

Idempotent addition defines a partial order: $x \leq y$ if and only if $x \oplus y = y$. We assume that this partial order extends to a linear order on $\mathbb{X}$.

For any $x \neq \mathbb{0}$ and integer $p > 0$, an integer power is defined in the usual way: $x^0 = \mathbb{1}$, $x^p = x^{p-1}x$, $x^{-p} = (x^{-1})^p$, $\mathbb{0}^p = \mathbb{0}$. It is assumed that the powers with rational exponents are also defined.

An example of the idempotent semifield is the real semifield $\mathbb{R}_{\max,+} = \langle \mathbb{R} \cup \{-\infty\}, -\infty, 0, \max, + \rangle$ for which $\mathbb{0} = -\infty$, $\mathbb{1} = 0$, $\oplus = \max$ and $\otimes = +$.

Let us denote by $\mathbb{X}^{m \times n}$ the set of matrices which consist of m rows and n columns with elements from $\mathbb{X}$. Addition and multiplication of conforming matrices $\boldsymbol{A} = (a_{ij})$, $\boldsymbol{B} = (b_{ij})$ and $\boldsymbol{C} = (c_{ij})$, as well as multiplication by a scalar x are defined by the formulas

$$\{\boldsymbol{A} \oplus \boldsymbol{B}\}_{ij} = a_{ij} \oplus b_{ij}, \quad \{\boldsymbol{B}\boldsymbol{C}\}_{ij} = \bigoplus_{k} b_{ik} c_{kj}, \quad \{x\boldsymbol{A}\}_{ij} = x a_{ij}.$$

The order relation given above is extended to matrices and is understood entrywise.

Consider square matrices in $\mathbb{X}^{n \times n}$. The matrix $\boldsymbol{I}$ with elements equal to $\mathbb{1}$ on the main diagonal and $\mathbb{0}$ outside it is the identity matrix.

For any square matrix $\boldsymbol{A} = (a_{ij})$ and integer $p > 0$ the power is given by: $\boldsymbol{A}^0 = \boldsymbol{I}$, $\boldsymbol{A}^p = \boldsymbol{A}^{p-1}\boldsymbol{A}$. We define the functions

$$\operatorname{tr} \boldsymbol{A} = \bigoplus_{i=1}^{n} a_{ii}, \quad \operatorname{Tr}(\boldsymbol{A}) = \bigoplus_{k=1}^{n} \operatorname{tr} \boldsymbol{A}^k,$$

If $\operatorname{Tr}(\boldsymbol{A}) \leq \mathbb{1}$, then the Kleene matrix is defined in the form

$$\boldsymbol{A}^* = \bigoplus_{k=0}^{n-1} \boldsymbol{A}^k.$$

The set of column vectors consisting of n elements is denoted by $\mathbb{X}^n$.

A vector without zero elements is called regular.

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For any nonzero vector $\boldsymbol{x} = (x_i) \in \mathbb{X}^n$, the transposed vector is denoted as $\boldsymbol{x}^T$. The multiplicatively conjugate vector for $\boldsymbol{x}$ is the row vector $\boldsymbol{x}^- = (x_i^-)$, where $x_i^- = x_i^{-1}$ if $x_i \neq \mathbb{0}$ and $x_i^- = \mathbb{0}$ – otherwise.

4. Solution of the optimal planning problem

Let us formulate the problem (1) in terms of the idempotent semifield $\mathbb{R}_{\max,+}$. We denote the following matrices and vectors:

$$\boldsymbol{B} = (b_{ij}), \quad \boldsymbol{C} = (c_{ij}), \quad \boldsymbol{D} = (d_{ij}),$$
$$\boldsymbol{x} = (x_i), \quad \boldsymbol{y} = (y_i), \quad \boldsymbol{f} = (f_i), \quad \boldsymbol{g} = (g_i), \quad \boldsymbol{h} = (h_i), \quad \boldsymbol{p} = (p_i).$$

The problem (1) in vector notation has the form

$$\begin{aligned} \min_{\boldsymbol{x},\boldsymbol{y}} \quad & \boldsymbol{x}^-\boldsymbol{p} \oplus \boldsymbol{p}^-\boldsymbol{x}, \\ & \boldsymbol{B}\boldsymbol{x} \leq \boldsymbol{x}, \quad \boldsymbol{C}\boldsymbol{x} = \boldsymbol{y}, \quad \boldsymbol{D}\boldsymbol{y} \leq \boldsymbol{x}, \quad \boldsymbol{g} \leq \boldsymbol{x} \leq \boldsymbol{h}, \quad \boldsymbol{y} \leq \boldsymbol{f} \end{aligned} \tag{2}$$

The solution of the problem is described by the following statement.

Lemma 1. *Let $\boldsymbol{B}$ and $\boldsymbol{D}$ be matrices, $\boldsymbol{C}$ be a column-regular matrix such that the matrix $\boldsymbol{R} = \boldsymbol{B} \oplus \boldsymbol{D}\boldsymbol{C}$ satisfies $\mathrm{Tr}(\boldsymbol{R}) \leq \mathbb{1}$. Let $\boldsymbol{g}$ be a vector, and $\boldsymbol{f}$ and $\boldsymbol{h}$ be regular vectors such that the vector $\boldsymbol{s}^T = \boldsymbol{f}^-\boldsymbol{C} \oplus \boldsymbol{h}^-$ satisfies the condition $\boldsymbol{s}^T\boldsymbol{R}^*\boldsymbol{g} \leq \mathbb{1}$.*

Then the minimum value of the objective function in problem (2) is equal to

$$\theta = (\boldsymbol{p}^-\boldsymbol{R}^*\boldsymbol{p})^{1/2} \oplus \boldsymbol{s}^T\boldsymbol{R}^*\boldsymbol{p} \oplus \boldsymbol{p}^-\boldsymbol{R}^*\boldsymbol{g},$$

and all regular solutions have the form

$$\boldsymbol{x} = \boldsymbol{R}^*\boldsymbol{u}, \quad \boldsymbol{y} = \boldsymbol{C}\boldsymbol{R}^*\boldsymbol{u}, \tag{3}$$

where $\boldsymbol{u}$ is any regular vector which satisfies the conditions

$$\boldsymbol{g} \oplus \theta^{-1}\boldsymbol{p} \leq \boldsymbol{u} \leq ((\boldsymbol{s}^T \oplus \theta^{-1}\boldsymbol{p}^-)\boldsymbol{R}^*)^-. \tag{4}$$

5. Conclusion

A project scheduling problem is considered, which consists in minimizing the maximum deviation of start times of jobs from given due dates under given constraints of the form “start-start”, “start-finish”, “finish-start” and boundaries for the earliest and latest allowed start time. A direct solution of the problem is obtained, which can be used for both formal analysis and direct calculations.

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Gubanov S. A., Design bureau "Luch", software engineer, segubanov@mail.ru