Fourier series summation and A.N. Krylov convergence acceleration in CAS

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Fourier Series summation

Since Fourier's book «Théorie analytique de la chaleur» (1822), Fourier series have been one of the irreplaceable forms of representation of solutions to the equations of mathematical physics. Solutions in the form of trigonometric series are regularly published in modern scientific periodicals: [Barseghyan, V.; Solodusha, S.6, Axioms 2022, 11, 157. MDPI], [Khromov A.P., 2021], [Beilin, 2011], [Dimovski, Spiridonova, 2011], [Kadyrov, Nasyrov, Suchkova, 2017], [Krupenin V, Dokl. URSS, 1990], etc. We consider the problem of summation of Fourier series in finite terms. We will investigate the Fourier series associated with the vibrations of a finite string using modern CAS.

Start point: simplest Green's function

$$\begin{cases} \dfrac{\partial^2 g}{\partial t^2} = \dfrac{\partial^2 g}{\partial x^2}, 0 < x < \pi, t > \tau, 0 \leq \tau < +\infty; \\ g|_{t=\tau} = 0, \ \left.\dfrac{\partial g}{\partial t}\right|_{t=\tau} = \delta(x-s), 0 < x < \pi, 0 < s < \pi; \\ g|_{x=0} = 0, \ g|_{x=l} = 0, \ \tau < t < +\infty \end{cases} \tag{1}$$

Here $\delta(x-s)$ is Dirac delta function. The Green's function g can be found as a Fourier series [Tikhonov, Samarskii]:

$$g = \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin n(t-\tau)\sin ns \sin nx. \tag{2}$$

This series has been studied by many authors, but is not presented in the available literature in finite terms.

Start point: simplest Green's function

Theorem

The Green's function g can be presented in finite terms as the expression:

$$g = \frac{1}{2\pi}\left(\operatorname{atan}\left(\cot\frac{t-\tau-x+s}{2}\right) + \operatorname{atan}\left(\cot\frac{t-\tau+x-s}{2}\right) - \right.$$
$$\left. - \operatorname{atan}\left(\cot\frac{t-\tau-x-s}{2}\right) - \operatorname{atan}\left(\cot\frac{x+s+t-\tau}{2}\right)\right)$$

Alternative expression has the form:

$$g = \frac{1}{2}\left(\left[\frac{x-s+T}{2\pi}\right] + \left[\frac{-x+s+T}{2\pi}\right] \right.$$
$$\left. - \left[\frac{-x-s+T}{2\pi}\right] - \left[\frac{x+s+T}{2\pi}\right]\right).$$

Here $[\cdot]$ means «floor», $T = t - \tau$.

Our Research

Leaving aside further mathematical studies of Green's functions, we fix the following questions:

- Can we find such kind of expressions in finite terms of Fourier series in moderm CAS?
- What kind or expressions for series can we find for Green's functions and other Fourier series?

There are some problems along the way. The most important of them is the impossibility of analytic continuation of Fourier series. This is important because modern CAS work with analytic functions of a complex variable by default. In the case of Fourier series there are arising interesting and unusual representations of sums in the form of some special functions. For the user, this can create difficulties, and we will discuss ways to overcome them.

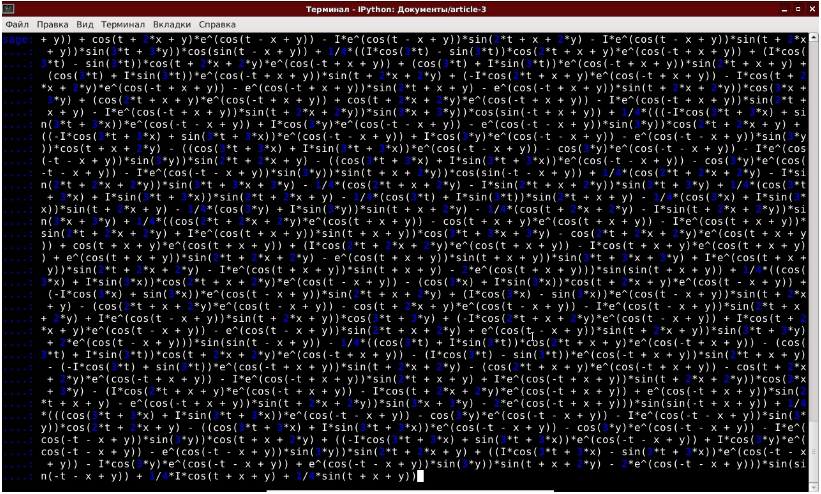
Fourier series summation, g

$$sum\left(\frac{1}{n}\cdot\sin(n\cdot x)\cdot\sin(n\cdot s)\cdot\sin(n\cdot t),\ n=1..\text{infinity}\right)\text{ assuming }x>0,\ s>0,\ t>0;$$

$$\frac{I}{8}\Big(\ln\big(1-e^{-I(t+x+s)}\big)-\ln\big(1-e^{-I(-t+x+s)}\big)-\ln\big(1-e^{I(-t-x+s)}\big)+\ln\big(1-e^{I(t-x+s)}\big)-\ln\big(1-e^{-I(t-x+s)}\big)+\ln\big(1-e^{-I(-t-x+s)}\big)+\ln\big(1-e^{I(-t+x+s)}\big)-\ln\big(1-e^{I(t+x+s)}\big)\Big)$$

The simplest Green's function g in finite terms in CAS Maple'2019. We can see satisfactory result, but too difficult for users. We cannot find this result without assuming, because this representation does not hold for arbitrary complex values of x, s, t.

Fourier series summation, g



The simplest Green's function g in finite terms in CAS Sage.

Another Green's function, $\tilde{g}$

Consider the series:

$$\tilde{g} = \frac{4}{\pi} \sum_{k=0}^{\infty} \frac{1}{2k+1} \sin\left(k + \frac{1}{2}\right) x \sin\left(k + \frac{1}{2}\right) s \sin\left(k + \frac{1}{2}\right) (t-\tau).$$

In finite terms:

$$\tilde{g} = \frac{1}{4} \left(\operatorname{sign} \sin \frac{x - s + t - \tau}{2} + \operatorname{sign} \sin \frac{-x + s + t - \tau}{2} + \right.$$
$$\left. \operatorname{sign} \sin \frac{x + s - t + \tau}{2} + \operatorname{sign} \sin \frac{-x - s - t + \tau}{2} \right).$$

This test turned out to be much more difficult.

Another Green's function, $\tilde{g}$

$$> sum\left(\frac{\sin\left(\left(n+\frac{1}{2}\right)x\right)\cdot\sin\left(\left(n+\frac{1}{2}\right)\cdot s\right)\cdot\sin\left(\left(n+\frac{1}{2}\right)\cdot t\right)}{\left(n+\frac{1}{2}\right)}, n=0..\text{infinity}\right) \text{ assuming } x>0, s>0, t>0$$

$$\Big(\tag{65}$$

$$-\frac{I}{4}\Big(\arctan\left(\sqrt{-e^{-I(s-x-t)}}\right)\sqrt{-e^{I(s+x+t)}}\sqrt{-e^{I(s+x-t)}}\sqrt{-e^{-I(s-x+t)}}\sqrt{-e^{I(s-x+t)}}\sqrt{-e^{I(s-x-t)}}\sqrt{-e^{-I(s+x-t)}}\sqrt{-e^{-I(s+x+t)}}$$

$$e^{-\frac{I}{2}(s-x-t)}$$

$$-\sqrt{-e^{-I(s-x-t)}}\Big(\sqrt{-e^{I(s+x+t)}}\sqrt{-e^{I(s+x-t)}}\sqrt{-e^{-I(s-x+t)}}\arctan\left(\sqrt{-e^{I(s-x-t)}}\right)e^{\frac{I}{2}(s-x-t)}\sqrt{-e^{I(s-x+t)}}\sqrt{-e^{-I(s+x-t)}}$$

$$\sqrt{-e^{-I(s+x+t)}}+\Big(e^{-\frac{I}{2}(s-x+t)}\sqrt{-e^{-I(s+x-t)}}\sqrt{-e^{-I(s+x+t)}}\sqrt{-e^{I(s-x+t)}}\sqrt{-e^{I(s+x-t)}}\sqrt{-e^{I(s+x+t)}}\arctan\left(\sqrt{-e^{-I(s-x+t)}}\right)$$

$$+\sqrt{-e^{-I(s-x+t)}}\Big(e^{-\frac{I}{2}(s+x-t)}\sqrt{-e^{-I(s+x+t)}}\sqrt{-e^{I(s-x+t)}}\sqrt{-e^{I(s+x-t)}}\sqrt{-e^{I(s+x+t)}}\arctan\left(\sqrt{-e^{-I(s+x-t)}}\right)$$

$$-\sqrt{-e^{-I(s+x-t)}}\Big(e^{\frac{I}{2}(s-x+t)}\sqrt{-e^{-I(s+x+t)}}\sqrt{-e^{I(s+x-t)}}\sqrt{-e^{I(s+x+t)}}\arctan\left(\sqrt{-e^{I(s-x+t)}}\right)$$

$$+\sqrt{-e^{I(s-x+t)}}\Big(e^{\frac{I}{2}(s+x-t)}\sqrt{-e^{-I(s+x+t)}}\sqrt{-e^{I(s+x+t)}}\arctan\left(\sqrt{-e^{I(s+x-t)}}\right)$$

$$+\sqrt{-e^{I(s+x-t)}}\Big(e^{-\frac{I}{2}(s+x+t)}\sqrt{-e^{I(s+x+t)}}\arctan\left(\sqrt{-e^{-I(s+x+t)}}\right)-e^{\frac{I}{2}(s+x+t)}\sqrt{-e^{-I(s+x+t)}}\arctan\left(\sqrt{-e^{I(s+x+t)}}\right)\Big)\Big)\Big)\Big)\Big)$$

$$\sqrt{-e^{I(s-x-t)}}\Big)\Big)\Big)\Big/$$

$$\left(\sqrt{-e^{-I(s-x-t)}}\sqrt{-e^{I(s+x+t)}}\sqrt{-e^{-I(s+x-t)}}\sqrt{-e^{-I(s-x+t)}}\sqrt{-e^{I(s-x-t)}}\sqrt{-e^{-I(s+x+t)}}\sqrt{-e^{I(s+x-t)}}\sqrt{-e^{I(s-x+t)}}\right)$$

Fourier testing results: tipycal series from textbook

This is the series for motion of finite string. It satisfies the initial condition $u(0,x) = \varphi(x) = x^2(1-x)$.

$$u = \sum_{n=1}^{\infty} \frac{8\cdot(-1)^{n+1}-4}{\pi^3 n^3} \sin(\pi n x)\cos(\pi n c t)$$

Maple'2019 is able to convert the infinite series in symbolic expression, namely

$$\begin{aligned} u = \frac{2i}{\pi^3}\Big(&\mathrm{Li}_3(-e^{i\cdot\pi(x+ct)}) - \mathrm{Li}_3(-e^{-i\cdot\pi(x+ct)}) + \\ &+ \mathrm{Li}_3(-e^{i\cdot\pi(x-ct)}) - \mathrm{Li}_3(-e^{-i\cdot\pi(x-ct)})\Big) - \\ &- \frac{i}{\pi^3}\Big(\mathrm{Li}_3(e^{-i\cdot\pi(x+ct)}) - \mathrm{Li}_3(e^{i\cdot\pi(x+ct)}) + \\ &+ \mathrm{Li}_3(e^{-i\cdot\pi(x-ct)}) - \mathrm{Li}_3(e^{i\cdot\pi(x-ct)})\Big). \end{aligned}$$

Here $\mathrm{Li}_3(z)$ is Euler's polylogarithm.

Fourier testing results: tipycal series from textbook

But $obviously$, at every moment of time it is a piecewise polynomial function! It is obviously, because we can convert the product $sin \cdot cos$ to the sum of sines. The symbolic expression of the function u in finite terms again requires the piecewise constructions sign or $\arctan(\cot)$. There is an alternative: work in the field of complex numbers $\mathbb{C}$ and use special functions, or work in the field of real numbers $\mathbb{R}$ and use piecewise elementary functions. The bridge between the two representations in finite terms is the Fourier series.

Fourier testing results: «hipergeometric» series

Hypergeometric functions arise when trying to present in finite terms the series describing forced oscillations in the absence of resonance.

> $sum\left(\frac{\sin((2\cdot k+1)\cdot x)}{(2\cdot k+1)\cdot((2\cdot k+1)^2-w^2)}, k=0..\text{infinity}\right)$ assuming $x>0$;

$$\frac{1}{2\,w^2-2}\left(-\mathrm{I}\,e^{-\mathrm{I}x}\,\mathrm{hypergeom}\left(\left[\frac{1}{2},1,\frac{w}{2}+\frac{1}{2},\frac{1}{2}-\frac{w}{2}\right],\left[\frac{3}{2},\frac{3}{2}+\frac{w}{2},\frac{3}{2}-\frac{w}{2}\right],e^{-2\mathrm{I}x}\right)+\mathrm{I}\,e^{\mathrm{I}x}\,\mathrm{hypergeom}\left(\left[\frac{1}{2},1,\frac{w}{2}+\frac{1}{2},\frac{1}{2}-\frac{w}{2}\right],\left[\frac{3}{2},\frac{3}{2}+\frac{w}{2},\frac{3}{2}-\frac{w}{2}\right],e^{2\mathrm{I}x}\right)\right) \quad \textbf{(45)}$$

> $sum\left(\frac{\sin((2\cdot k+1)\cdot x)\cdot\sin((2\cdot k+1)\cdot y)}{(2\cdot k+1)^2\cdot((2\cdot k+1)^2-w^2)}, k=0..\text{infinity}\right)$ assuming $x>0, y>0$;

$$\frac{1}{4\,w^2-4}\left(-\mathrm{hypergeom}\left(\left[\frac{1}{2},\frac{1}{2},1,\frac{w}{2}+\frac{1}{2},\frac{1}{2}-\frac{w}{2}\right],\left[\frac{3}{2},\frac{3}{2},\frac{3}{2}+\frac{w}{2},\frac{3}{2}-\frac{w}{2}\right],e^{2\mathrm{I}(x-y)}\right)e^{\mathrm{I}(x-y)}-\mathrm{hypergeom}\left(\left[\frac{1}{2},\frac{1}{2},1,\frac{w}{2}+\frac{1}{2},\frac{1}{2}-\frac{w}{2}\right],\left[\frac{3}{2},\frac{3}{2},\frac{3}{2}+\frac{w}{2},\frac{3}{2}-\frac{w}{2}\right],e^{-2\mathrm{I}(x-y)}\right)e^{-\mathrm{I}(x-y)}+e^{-\mathrm{I}(x+y)}\,\mathrm{hypergeom}\left(\left[\frac{1}{2},\frac{1}{2},1,\frac{w}{2}+\frac{1}{2},\frac{1}{2}-\frac{w}{2}\right],\left[\frac{3}{2},\frac{3}{2},\frac{3}{2}+\frac{w}{2},\frac{3}{2}-\frac{w}{2}\right],e^{-2\mathrm{I}(x+y)}\right)+\mathrm{hypergeom}\left(\left[\frac{1}{2},\frac{1}{2},1,\frac{w}{2}+\frac{1}{2},\frac{1}{2}-\frac{w}{2}\right],\left[\frac{3}{2},\frac{3}{2},\frac{3}{2}+\frac{w}{2},\frac{3}{2}-\frac{w}{2}\right],e^{2\mathrm{I}(x+y)}\right)e^{\mathrm{I}(x+y)}\right) \quad \textbf{(46)}$$

Fourier testing results: Lerch's Zeta-function

> $sum\left(\frac{(-1)^{k-1}}{w^2-k^2}\cdot\sin(k\cdot x)\cdot\sin(k\cdot T),\ k=0..\text{infinity}\right)$ assuming $x>0,\ X>0,\ T>0,\ w>0;$

$$\frac{1}{8\,w}\Big(\mathrm{LerchPhi}(-e^{I(x+T)},1,w)-\mathrm{LerchPhi}(-e^{I(x+T)},1,-w)-\mathrm{LerchPhi}(-e^{-I(-x+T)},1,w)+\mathrm{LerchPhi}(-e^{-I(-x+T)},1,-w)-\mathrm{LerchPhi}(-e^{I(-x+T)},1,w)+\mathrm{LerchPhi}(-e^{I(-x+T)},1,-w)+\mathrm{LerchPhi}(-e^{-I(x+T)},1,w)-\mathrm{LerchPhi}(-e^{-I(x+T)},1,-w)\Big) \quad \textbf{(49)}$$

Here Lerch's transcendent is the series

$$\Phi(z,s,v)=\sum_{n=0}^{\infty}\frac{z^n}{(v+n)^s}.$$

See also [V. M. Kaplitskii, "A differential equation for Lerch's transcendent and associated symmetric operators in Hilbert space", Sb. Math., 205:8 (2014)].

Conclusions from experiments

- The application of standard algorithms for the summation of Fourier series can be difficult using the field of complex numbers
- The relationship between piecewise functions built into CAS («floor», «sign») and elementary functions in the sense of Liouville theory is either not written in CAS or has a non-trivial implementation that is unknown to us
- Systematized special functions used in CAS for summation trigonometric Fourier series in finite terms

A.N. Krylov's technique Fourier series

The direct application of CAS to the summation of Fourier series can lead to difficulties.

One can try to overcome them by changing the formulation of the problem: instead of the summation problem in the finite terms, consider the problem of accelerating the convergence of the Fourier series. As Krylov wrote, this technique «often leads to the representation of the sum of the proposed series in closed form under the guise piecewice function».

A.N. Krylov's method and basic series

$$\sum_{i=1}^{\infty}\left(U\left(\frac{1}{n}\right)\sin nx+V\left(\frac{1}{n}\right)\cos nx\right)$$

$$U\left(\frac{1}{n}\right)=u_1\frac{1}{n}+u_2\frac{1}{n^2}+O\left(\frac{1}{n^3}\right),u_1,u_2\in\mathbb{R};$$

$$V\left(\frac{1}{n}\right)=v_1\frac{1}{n}+v_2\frac{1}{n^2}+O\left(\frac{1}{n^3}\right),v_1,v_2\in\mathbb{R};$$

$$\sum_{n=1}^{\infty}\frac{\sin nx}{n}=\frac{\pi-x}{2},x\in(0,2\pi]$$

$$\sum_{n=1}^{\infty}\frac{\cos nx}{n}=-\log 2\left|\sin\frac{x}{2}\right|,x\in(0,2\pi)$$

We can see, that the simplest Green's function g is the subject, when this scheme has only ONE nontrivial step.

Continuation of basic series

Reconstruction

$$\sum_{n=1}^{\infty} \frac{8\cdot(-1)^{n+1}-4}{\pi^3 n^3}\sin(\pi n y) = ??$$

$$s1 := \sum_{n=1}^{\infty} \frac{\sin nx}{n^3} = sgn(\cos\frac{x}{4})\left\{\frac{t^3}{12}+\frac{\pi^2 t}{6}-\frac{\pi t|t|}{4}\right\}\Big|_{t=4\mathrm{asin}\,\sin\frac{x}{4}}$$

$$s2 := \sum_{n=1}^{\infty} (-1)^n \frac{\sin nx}{n^3} = s1|_{x-\pi}.$$

Krylov's technique in moderm mathemathical physics

- А. Б. Нерсесян "Ускорение сходимости разложений по собственным функциям", Докл. НАН Армении.2007.
- А. П. Хромов, М. Ш. Бурлуцкая, "Классическое решение методом Фурье смешанных задач при минимальных требованиях на исходные данные", Изв. Сарат. ун-та. Нов. сер. Сер. Математика. Механика. Информатика, 14:2 (2014), 171–198
- Adcock B. Modified Fourier expansions: theory, construction and application / Trinity Hall University of Cambridge, 2010.

Results presented in the talk

1. The representation of the Green's functions of finite strings in finite terms. It is possible for some class of boundary conditions, including Dirichlet and mixed boundary conditions
2. The experiments for summation Fourier series in Computer Algebra Systems
3. Some difficulties for implementation A.N.Krylov's method in CAS

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Thank you for your time and attention!